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UTAH UNIV SALT LAKE CITY DEPT OF MATHEMATICS  
AN ALGORITHM FOR THE ELECTROMAGNETIC SCATTERING FROM AN AXIALLY--ETC(U)  
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AN ALGORITHM FOR THE ELECTROMAGNETIC SCATTERING DUE TO AN  
AXIALLY SYMMETRIC BODY WITH AN IMPEDANCE BOUNDARY CONDITION\*

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## ABSTRACT

Let  $B$  be a body in  $R^3$ , and let  $S$  denote the boundary of  $B$ . The surface  $S$  is described by  $S = \{(x,y,z) : \sqrt{x^2 + y^2} = f(z), -1 \leq z \leq 1\}$  where  $f$  is an analytic function that is real and positive on  $(-1,1)$  and  $f(\pm 1) = 0$ . An algorithm is described for computing the scattered field due to a plane wave incident field, under Leontovich boundary conditions. The Galerkin method of solution used here leads to a block diagonal matrix involving  $2M + 1$  blocks, each block being of order  $2(2N+1)$ . If e.g.  $N = O(M^2)$ , the computed scattered field is accurate to within an error bounded by  $Ce^{-cN^{1/2}}$  where  $C$  and  $c$  are positive constants depending only on  $f$ .

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1. INTRODUCTION AND SUMMARY

Let  $B$  be a bounded body in  $R^3$ , having surface  $S$  which is given by

$$(1.1) \quad S = \{(x, y, z) : \sqrt{x^2 + y^2} = f(z), -1 \leq z \leq 1\},$$

where  $f$  is an analytic function that is real and positive on  $(-1, 1)$  and

$$(1.2) \quad c_1 (1+z)^{\alpha_1} (1-z)^{\beta_1} \leq f(z) \leq c_2 (1+z)^{\alpha_2} (1-z)^{\beta_2}, -1 \leq z \leq 1,$$

where  $c_j$ ,  $\alpha_j < 1$  and  $\beta_j < 1$  are positive constants. In this paper we describe an algorithm for computing the field scattered from  $B$  due to an incident field  $\bar{E}^0(\bar{r})$  of the form

$$(1.3) \quad \bar{E}^0(\bar{r}) = \bar{\epsilon} e^{i\bar{k}_0 \cdot \bar{r}}$$

where  $\bar{\epsilon}$  and  $\bar{k}_0$  ( $|\bar{k}_0| = k_0 = \omega/c = 2\pi/\lambda$ ) denote the polarization and propagation vectors respectively, and  $\bar{r} = x\hat{x} + y\hat{y} + z\hat{z}$ , where  $\hat{x}$ ,  $\hat{y}$  and  $\hat{z}$  are the unit vectors pointing in the direction of the  $x$ ,  $y$  and  $z$  axes respectively.

Let the body  $B$  (resp. free space) be homogeneous, with permittivity  $\epsilon$  (resp.  $\epsilon_0$ ) permeability  $\mu$  (resp.  $\mu_0$ ) and conductivity  $\sigma$  (resp.  $\sigma_0$ ), so that the refractive index of the body is

$$(1.4) \quad N = \left\{ \frac{\mu}{\mu_0} \left( \frac{\epsilon}{\epsilon_0} + \frac{i\sigma}{\omega\epsilon_0} \right) \right\}^{1/2}$$



where  $\omega$  denotes the frequency of the incident field. We shall furthermore assume that

$$(1.5) \quad |N| |k_0| \rho \gg 1,$$

where  $\rho$  is the smallest radius of curvature of  $S$ . This assumption enables us to apply the Leontovich boundary conditions [15, 21]

$$(1.6) \quad (\hat{n} \times \bar{E}) \times \hat{n} = \eta Z \hat{n} \times \bar{H}$$

on the surface of the body, where

$$(1.7) \quad \eta = \mu/(\mu_0 N), \quad Z = (\mu_0 / \epsilon_0)^{1/2}$$

and where  $\hat{n}$  denotes the outward unit normal to  $S$ , and  $\bar{E}$  and  $\bar{H}$  denote the total electric and magnetic fields on  $S$ . The conditions (1.5) and (1.6) are satisfied automatically if the body  $B$  is perfectly conducting, i.e. if  $\sigma = \infty$ .

The condition (1.6) makes it possible to obtain a singular vector integral equation over  $S$  for the surface current  $\bar{K}$  on  $S$ . In the present paper we describe an algorithm for solving this integral equation via the Galerkin method, using as basis functions

$$(1.8) \quad \psi_{mn}(z, \varphi) = e^{im\varphi} f^{1/2}(z) (1-z)^2 \frac{\sin[\pi N^{1/2} \{ \log(\frac{1+z}{1-z}) - \frac{n}{N^{1/2}} \} ]}{\pi N^{1/2} \{ \log(\frac{1+z}{1-z}) - \frac{n}{N^{1/2}} \}}$$

$$m = 0, \pm 1, \dots, \pm N, \quad n = 0, \pm 1, \dots, \pm M.$$

These basis functions effectively handle singularities of  $\bar{K}$  and  $\frac{\partial \bar{K}}{\partial z}$  as a function of  $z$ , which occur at  $z = \pm 1$ ; they are similarly very effective approximants in the  $\phi$  variable, since the Fourier series of  $\bar{E}^0$  (and therefore  $\bar{K}$ ) converges very rapidly. The singularities occurring in the kernel of the integral equation for  $\bar{K}$  are of the type  $1/(z' - z)$  or  $\log |z' - z|$  at  $z = z'$ , and of the type of  $f^m(z)$ ,  $m = 0, 1, \dots$  at  $z = \pm 1$ . The first of these is effectively handled by subtracting out the principal value. The remaining ones are effectively handled by means of the quadrature formula (see [22])

$$(1.9) \quad \int_{-1}^1 g(t) dt \approx h \sum_{k=-L}^L \frac{2e^{kh}}{(1+e^{kh})^2} g\left(\frac{e^{kh}-1}{e^{kh}+1}\right)$$

after transforming the intervals  $(-1, z)$  and  $(z, 1)$  to  $(-1, 1)$ .

The integrations over  $S$  involve two variables,  $\phi'$  and  $z'$ . While the integrations with respect to  $z'$  must be carried out numerically, due to singularities of the kernel in the region of integration, the integrations with respect to  $\phi$  are carried out explicitly, the results being expressed via hypergeometric functions. The hypergeometric functions have logarithmic singularities which were not present in the kernel; for this reason explicit integration and later evaluation of the hypergeometric function as described in the Appendix has an advantage over direct numerical integration, since any known direct numerical integration procedure such as the trapezoidal rule would poorly handle this type of transformation of singularity.

The use of the basis function (1.8) thus leads to a block diagonal Galerkin system of equations, one system of order  $2(2N+1)$  for each  $m$ . By forming  $2M+1$  such blocks, and taking  $N=M^2$ , we arrive an approximation  $\bar{K}_N$  of  $\bar{K}$  which is accurate to within an error  $\bar{\epsilon}$ , where  $|\bar{\epsilon}| = O(e^{-cN^{1/2}})$  as  $N \rightarrow \infty$  with  $c > 0$  and independent of  $N$ . The use of (1.8) furthermore makes it possible to evaluate the scattered field  $\bar{E}^s$  by means of simple one-dimensional trapezoidal rule integrations. The error  $\bar{\delta}_N$  in our approximation  $\bar{E}_N^s$  of the scattered field  $\bar{E}^s$  satisfies the relation

$$(1.10) \quad |\bar{\delta}_N(\bar{r})| = |\bar{E}^S(\bar{r}) - \bar{E}_N^S(\bar{r})| = O(e^{-cN^{1/2}}) \text{ as } N \rightarrow \infty,$$

for all  $\bar{r}$  on the exterior of  $B$  and not arbitrarily close to  $B$ .

The above problem of computing  $\bar{E}^S$  given  $\bar{E}^0$  as in (1.2) and  $S$  as in (1.1) was studied in [20] for the perfectly conducting case and in [5] for the case as described above. The Galerkin approximating basis function used in [5,20] are of the form  $\psi_{mn} = \cos(m\phi) S_n(z)$  and  $\sin(m\phi) S_n(z)$ , where  $S_n$  is the linear spline which is zero at  $z_{n-1}$  and  $z_{n+1}$ , and 1 at  $z_n$ . Thus the resulting rate of convergence is  $O(1/N^2)$  if  $f$  has no singularities at  $z = \pm 1$  and  $O(1/N^\alpha)$  if e.g.  $f(z) \sim C(1-z)^\alpha$  as  $z \rightarrow 1$ , where it is assumed that the interval  $(-1,1)$  is divided into  $N$  subintervals. In addition, the quadratures used in [5,20] converge very slowly as a result of the singularities present in the integral equation. Furthermore the expression for the gradient of  $G = e^{ik_0|\bar{r}-\bar{r}'|}/(4\pi|\bar{r}-\bar{r}'|)$  obtained in [5,20] is incorrect.

The algorithm of the present paper has been checked out on a computer for the case of a sphere of radius 1 for which the surface current  $\bar{K}$  and the scattered field  $\bar{E}^S$  can be expressed explicitly. Using  $M = 4$  and  $N=M^2 = 6$ , we find that  $\bar{K}$  is accurate to 4 significant figures, and for  $r \geq 2$ ,  $\bar{E}^S$  is also accurate to 4 significant figures.

The paper is organized as follows.

In Sec. 2 we describe the geometry of the surface. In Sec. 3 we derive a representation on the surface  $S$  for the incident plane wave. Sec. 4 contains a derivation of the integral equation for the surface current, as well as an integral expression for the scattered field in terms of this surface current. In Sec. 5 we describe the basis functions



to be used in the Galerkin method of Sec. 7. Sec. 6 contains an approximate representation of the incident electric field, in terms of the basis functions of Sec. 5. In Sec. 7 we derive the Galerkin equations for the surface current, and we describe a method of computing the coefficients of this system, and for solving this system. In Sec. 8 we describe a procedure for evaluating the scattered field. In Sec. 9 we discuss the rate convergence of the procedure. Finally, Sec. 10 contains a numerical example, illustrating the algorithm. Appendix A contains a study of the functions  $G_m$  derived in Sec. 7 as well as of their derivatives. The results of this appendix illustrate the type of singular behavior of the functions  $G_m$  and thus they dictate the type of approximate methods to be used in order to achieve high accuracy, and they simplify our proof of convergence.

The rate of convergence of the method of this paper, namely  $O(e^{-cN^{1/2}})$  using an approximation of the form

$$(1.11) \quad \sum_{m=-M}^M \sum_{n=-N}^N a_{mn} \theta_n(z) e^{im\varphi} \quad (N=M^2)$$

for each component of the surface current is optimal, in a certain sense. By the results of [25], given any approximation method of the type (1.11) which is to converge for all  $f$  analytic on  $(-1,1)$  and satisfying (1.2), the resulting error of this method cannot converge to zero faster than  $e^{-\gamma N^{1/2}}$ , for some  $\gamma > 0$ .

Similar numerical methods have been considered but it is believed that the order of convergence obtained is not as good as that demonstrated here. For further discussion of such methods see for example Andreassen [2] whose proposed method considers a maximum period of 20



wave lengths. Barber and Yeh [ 3 ] and Waterman [ 27 ] have considered extended boundary methods, while Kennaugh [ 13 ] and Schultz, et. al. [ 19 ] have discussed other implementations using a product  $z, \phi$  basis.

## 2. GEOMETRY OF THE SURFACE

A point on the surface  $S$  is represented by

$$(2.1) \quad \bar{r} = f(z) \cos\varphi \hat{x} + f(z) \sin\varphi \hat{y} + z\hat{z},$$

where  $\hat{x}$ ,  $\hat{y}$  and  $\hat{z}$  denote the unit vectors in the direction of the  $x$ ,  $y$  and  $z$  axes respectively.

It is convenient to introduce three unit vectors on the surface,  $\hat{n}$ ,  $\hat{\varphi}$  and  $\hat{t}$  where:

$$(2.2) \quad \begin{aligned} \hat{n} &= \alpha(z) \cos\varphi \hat{x} + \alpha(z) \sin\varphi \hat{y} - f'(z)\alpha(z)\hat{z} \\ \hat{\varphi} &= -\sin\varphi \hat{x} + \cos\varphi \hat{y} \\ \hat{t} &= f'(z)\alpha(z) \cos\varphi \hat{x} + f'(z)\alpha(z) \sin\varphi \hat{y} + \alpha(z)\hat{z}, \end{aligned}$$

where;

$$(2.3) \quad \alpha(z) = [1 + f'(z)^2]^{-1/2}.$$

The vector  $\hat{n}$  is the unit normal to the surface,  $\hat{\varphi}$  is the unit vector at  $\bar{r}$ , pointing in the direction of increasing  $\varphi$ , and  $\hat{t}$  is the unit longitudinal vector, pointing in the direction of increasing arc length. Thus  $\hat{n}$ ,  $\hat{\varphi}$ ,  $\hat{t}$  and  $\hat{x}$ ,  $\hat{y}$ ,  $\hat{z}$  are related by means of the equations

$$(2.4) \quad \begin{pmatrix} \hat{n} \\ \hat{\varphi} \\ \hat{t} \end{pmatrix} = \begin{pmatrix} \alpha(z) \cos \varphi & \alpha(z) \sin \varphi & -f'(z) \alpha(z) \\ -\sin \varphi & \cos \varphi & 0 \\ f'(z) \alpha(z) & \cos \varphi f'(z) \alpha(z) & \sin \varphi \alpha(z) \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

and

$$(2.5) \quad \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \alpha(z) \cos \varphi & -\sin \varphi & f'(z) \alpha(z) \cos \varphi \\ \alpha(z) \sin \varphi & \cos \varphi & f'(z) \alpha(z) \sin \varphi \\ -f'(z) \alpha(z) & 0 & \alpha(z) \end{pmatrix} \begin{pmatrix} \hat{n} \\ \hat{\varphi} \\ \hat{t} \end{pmatrix}$$

### 3. THE INCIDENT ELECTRIC FIELD

Let the incident radiation be a plane wave, given by

$$(3.1) \quad E^0(\vec{r}) = \vec{\epsilon} e^{i\vec{k}_0 \cdot \vec{r}},$$

where  $\vec{\epsilon}$  points in the direction of polarization, and  $\vec{k}_0$  is the propagation vector which satisfies the relations

$$(3.2) \quad k_0 \equiv |\vec{k}_0| = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

$$\vec{k}_0 \cdot \vec{\epsilon} = 0$$

We shall furthermore assume that  $\vec{k}_0$  lies in the  $xz$  plane and makes an angle  $\theta_0$  with the  $z$ -axis. Thus

$$(3.3) \quad \vec{k}_0 = k_0 \sin \theta_0 \hat{x} + k_0 \cos \theta_0 \hat{z}.$$

It is convenient to set

$$(3.4) \quad \vec{\epsilon} = a_1 \vec{\epsilon}_1 + a_2 \vec{\epsilon}_2$$

where  $a_1$  and  $a_2$  are scalars, while

$$(3.5) \quad \vec{\epsilon}_1 = \hat{y}, \quad \vec{\epsilon}_2 = -\hat{x} \cos \theta_0 + \hat{z} \sin \theta_0.$$



Borison [4] has shown that if  $a_2 = 0$  (rsp.  $a_1 = 0$ ) then the back scattered electric field  $E^s(\bar{r})$  is polarized only in the direction  $\bar{e}_1$  (rsp.  $\bar{e}_2$ ).

Using (2.5) and (3.5) we can express  $\bar{E}$  in components of  $\hat{\varphi}$ ,  $\hat{t}$  and  $\hat{n}$ . We get

$$\begin{aligned} \bar{E} = & a_1 [\alpha(z) \sin \varphi \hat{n} + \cos \varphi \hat{\varphi} + \alpha(z) f'(z) \sin \varphi \hat{t}] \\ (3.6) \quad & + a_2 [-\alpha(z) (\cos \theta_0 \cos \varphi + \sin \theta_0 f'(z)) \hat{n} + \cos \theta_0 \sin \varphi \hat{\varphi} \\ & - \alpha(z) (\cos \theta_0 \cos f'(z) - \sin \theta_0) \hat{t}]. \end{aligned}$$

Next, let us find the Fourier expansion of  $E^0(\bar{r})$  on  $S$ . To this end we use the identity

$$(3.7) \quad e^{ix \cos \varphi} = \sum_{m=-\infty}^{\infty} i^m J_m(x) e^{im\varphi}$$

where  $J_m(x)$  denotes the Bessel function,  $J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2/4)^{m+n}}{n! (n+m)!}$ .

Using (2.1) and (3.3) we get

$$(3.8) \quad \bar{k}_0 \cdot \bar{r} = k_0 f(z) \sin \theta_0 \cos \varphi + k_0 z \cos \theta_0.$$

Hence combining (3.1), (3.7) and (3.8) we find that the incident field on  $S$  is given by

$$(3.9) \quad \bar{E}^0(\bar{r}) = \bar{E} e^{ik_0 z \cos \theta_0} \sum_{m=-\infty}^{\infty} i^m J_m(k_0 f(z) \sin \theta_0) e^{im\varphi}.$$

where  $\bar{E}$  is given in (3.6). Combining (3.6) and (3.9) we get

$$\begin{aligned}
 \bar{E}^0(\bar{r}) = & \{ [ a_1 \alpha(z) \sin \varphi - a_2 \alpha(z) (\cos \theta_0 \cos \varphi + f'(z) \sin \theta_0) ] \hat{n} \\
 & + [ a_1 \cos \varphi + a_2 \cos \theta_0 \sin \varphi ] \hat{\varphi} \\
 (3.10) \quad & + [ a_1 \alpha(z) f'(z) - a_2 \alpha(z) (\cos \theta_0 \cos \varphi f'(z) - \sin \theta_0) ] \hat{t} \} \\
 & \cdot e^{ik_0 z \cos \theta_0} \sum_{m=-\infty}^{\infty} i^m J_m(k_0 f(z) \sin \theta_0) e^{im\varphi}
 \end{aligned}$$

#### 4. THE SCATTERED FIELD AND THE INTEGRAL EQUATION FOR THE SURFACE CURRENT

The scattered field  $\bar{E}^s = \bar{E}^s(\bar{r})$  is given in terms of the total electric ( $\bar{E}$ ) and magnetic ( $\bar{H}$ ) fields on  $S$  by [26]

$$(4.1) \quad \bar{E}^s = - \int_S [ i\omega\mu(\hat{n} \times \bar{H})G + (\hat{n} \times \bar{E}) \times \nabla G + (\hat{n} \cdot \bar{E}) \nabla G ] dS$$

where it is assumed that the field vectors  $\bar{E}$  and  $\bar{H}$  have time dependence of the factored form  $e^{-i\omega t}$ , and

$$(4.2) \quad G = G(\bar{r}, \bar{r}') = \frac{1}{4\pi} \frac{\exp [ ik_0 |\bar{r} - \bar{r}'| ] }{|\bar{r} - \bar{r}'|} .$$

The remaining vectors in the integrand (4.1) are expressed in terms of  $\bar{r}$ , and  $\nabla$  is expressed in terms of  $\bar{r}$ .

Let  $\bar{K}$  denote the surface current on  $S$ , due to the fields  $\bar{E}$  and  $\bar{H}$ . In view of the Leontovich boundary condition (1.4),  $\bar{K}$  satisfies the relations

$$(4.3) \quad \begin{aligned} \bar{K} &= - \hat{n} \times \bar{H} ; \quad \hat{n} \times \bar{E} = - \eta Z \hat{n} \times \bar{K} \\ \sigma &= - \epsilon \hat{n} \cdot \bar{E} = \frac{1}{i\omega} \nabla \cdot \bar{K} \end{aligned}$$

Our development of the integral equation for  $\bar{K}$  follows that in [5]. We include the derivation in [5] for this equation, for sake of completeness.

Let  $\bar{A}$  be a continuous vector field tangent to  $S$ . Then the following results are valid, if  $S$  satisfies the Lyapunov conditions [17, p. 90] (a) the integral

$$I(\bar{r}') \equiv \int_S \bar{A}(\bar{r}) G(\bar{r}, \bar{r}') dS$$

is a continuous function of  $\bar{r}'$  in  $R^3$ .

(b) As  $\bar{r}' \rightarrow \bar{r}'_0 \notin S$ , the relations

$$\begin{aligned} & \hat{n}(\bar{r}'_0) \times \lim_{\bar{r}' \rightarrow \bar{r}'_0} \int_S \bar{A}(\bar{r}) \times \nabla G(\bar{r}, \bar{r}') dS \\ &= \pm \frac{1}{2} \bar{A}(\bar{r}'_0) + \int_S \hat{n}(\bar{r}'_0) \times [\bar{A}(\bar{r}) \times \nabla G(\bar{r}_0, \bar{r})] dS \end{aligned}$$

are satisfied where the plus (resp. minus) sign corresponds to an approach from the outside (resp. inside) of  $B$ .

(c) The term [17, p. 95]

$$\hat{n}(\bar{r}') \cdot \int_S \bar{A}(\bar{r}) \times \nabla G(\bar{r}, \bar{r}') dS$$

is a continuous function of  $\bar{r}'$  on  $S$ . The term

$$\bar{E}_3(\bar{r}') \equiv \int_S \hat{n}(\bar{r}) \cdot \bar{E}(\bar{r}) \nabla G(\bar{r}, \bar{r}') dS$$

suffers a discontinuity on transition through  $S$  equal to  $\hat{n} \cdot \Delta \bar{E}_3$ , where  $\Delta \bar{E}_3$  is the difference of the values outside and inside. The third term in  $\bar{E}^s$ , therefore does not affect the tangential component, but reduces the normal component of  $\bar{E}$  to zero.

Since the total electric field  $\bar{E} = \bar{E}^o + \bar{E}^s$  is zero inside the



scatterer, (4.1) yields

$$0 = \hat{n} \times \bar{E}^0 - \lim_{\bar{r}' \rightarrow \bar{r}'_0 \in S} \hat{n}(\bar{r}'_0) \times \int_S [i\omega\mu(\hat{n} \times \bar{H})G + (\hat{n} \times \bar{E}) \times \nabla G + (\hat{n} \cdot \bar{E}) \nabla G] dS$$

where the approach is from the inside.

Application of (b) gives

$$(4.4) \quad 0 = \hat{n}(\bar{r}') \times \bar{E}^0(\bar{r}') + \frac{1}{2} \hat{n}(\bar{r}') \times \bar{E}(\bar{r}') - \hat{n}(\bar{r}') \times \int_S [i\omega\mu(\hat{n} \times \bar{H})G + (\hat{n} \times \bar{E}) \times \nabla G + (\hat{n} \cdot \bar{E}) \nabla G] dS$$

Next, taking the limit as  $\bar{r}' \rightarrow \bar{r}'_0$  from the outside of B in (4.1) and using the relation

$$\hat{n}(\bar{r}'_0) \times \bar{E}(\bar{r}'_0) = \hat{n}(\bar{r}'_0) \times \{\bar{E}^0(\bar{r}'_0) + \bar{E}^s(\bar{r}'_0)\}$$

we get

$$\hat{n} \times \bar{E} = \hat{n} \times \bar{E}^0 - \lim_{\bar{r}' \rightarrow \bar{r}'_0} \hat{n}(\bar{r}'_0) \times \int_S [i\omega\mu(\hat{n} \times \bar{H})G + (\hat{n} \times \bar{E}) \times \nabla G + (\hat{n} \cdot \bar{E}) \nabla G] dS$$

Application of (b) yields

$$(4.5) \quad \frac{3}{2} \hat{n}(\bar{r}') \times \bar{E}(\bar{r}') = \hat{n}(\bar{r}') \times \bar{E}^0(\bar{r}') - \hat{n}(\bar{r}') \times \int_S [i\omega\mu(\hat{n} \times \bar{H})G + (\hat{n} \times \bar{E}) \times \nabla G + (\hat{n} \cdot \bar{E}) \nabla G] dS$$

Adding (4.4) and (4.5), we get

$$\hat{n}(\bar{r}') \times \bar{E}(\bar{r}') + 2\hat{n}(\bar{r}') \times \int_S [i\omega\mu(\hat{n} \times \bar{H})G + (\hat{n} \times \bar{E}) \times \nabla G + (\hat{n} \cdot \bar{E}) \nabla G] dS = 2\hat{n}(\bar{r}') \times \bar{E}^0(\bar{r})$$

The definitions (4.3) now yield

$$- \eta Z \hat{n}(\bar{r}') \times \bar{K}(\bar{r}') + 2\hat{n}(\bar{r}') \times \int_S \{ -i\omega\mu \bar{K}G - \eta Z(\hat{n} \times \bar{K}) \times \nabla G - \frac{1}{i\omega\epsilon} (\nabla \cdot \bar{K}) \nabla G \} dS = 2\hat{n}(\bar{r}') \times \bar{E}^0(\bar{r}')$$

This equation can be written in equivalent form

$$(4.6) \quad \left[ \frac{1}{2\eta} Z \bar{K}(\bar{r}') + \int_S [i\omega\mu \bar{K}G + Z(\hat{n} \times \bar{K}) \times \nabla G + \frac{1}{i\omega\epsilon} (\nabla \cdot \bar{K}) \nabla G] dS + \bar{E}^0(\bar{r}') \right]_{\tan \frac{1}{r} \in S} = 0$$

Using Eqs. (4.3), the scattered field  $\bar{E}^S$  given by (4.1) is expressed in terms of  $\bar{K}$  by means of the integral

$$(4.7) \quad \bar{E}^S = \int_S [i\omega\mu \bar{K}G + \eta Z(\hat{n} \times \bar{K}) \times \nabla G + \frac{1}{i\omega\epsilon} (\nabla \cdot \bar{K}) \nabla G] dS$$

# 5. THE BASIS FUNCTIONS FOR APPROXIMATING SURFACE CURRENT AND ELECTRIC FIELD

Let  $d > 0$  and  $d' > 1$ , and let  $\Omega_d$  and  $A_{d'}$  be defined by

$$(5.1) \quad \begin{aligned} \Omega_d &= \{ z \in \mathbb{C} : \left| \arg \left[ \frac{(1+z)}{(1-z)} \right] \right| < d \} \\ A_{d'} &= \{ w \in \mathbb{C} : 1/d' < |w| < d' \} \end{aligned}$$

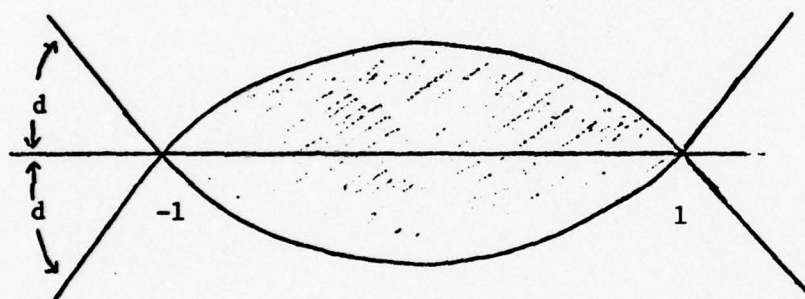


Figure 5.1. The Region  $\Omega_d$

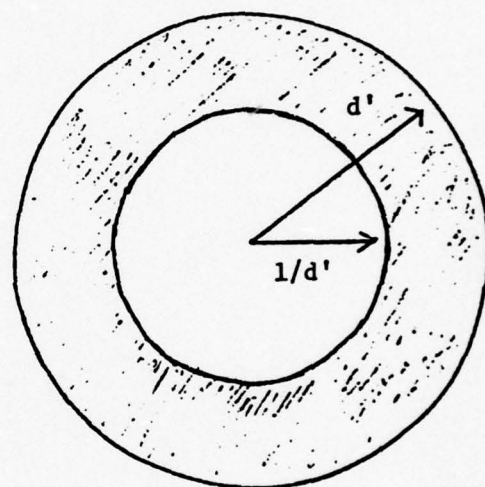


Figure 5.2. The Region  $A_{d'}$

Let  $H(\Omega_d)$  (rsp.  $H(A_{d'})$ ) denote the family of all functions  $g$  that are analytic in  $\Omega_d$  (rsp.  $A_{d'}$ ) such that

$$(5.2) \quad \|g\|_{\Omega_d} \equiv \sup_{z \in \Omega_d} |g(z)| < \infty \quad (\text{rsp. } \|g\|_{A_{d'}} = \sup_{z \in A_{d'}} |g(z)| < \infty).$$

Let us set

$$(5.3) \quad v(z) = f(z) (1-z^2)$$

where  $f \in H(\Omega_d)$  satisfies (1.2),  $f \neq 0$  in  $\Omega_d$ . Let  $H = H(d, d')$  denote the family of all functions  $F = F(z, w)$  such that

- i)  $F(z, e^{i\varphi}) / v(z)$  belongs to  $H(\Omega_d)$  as a function of  $z$  for all  $\varphi \in [0, 2\pi]$ ;
- ii)  $F(z, w)$  belongs to  $H(A_{d'})$  as a function of  $w$  for all  $z \in [-1, 1]$ .

We define a norm on  $H(d, d')$  by

$$(5.4) \quad \|F\|_H = \max \left\{ \max_{\varphi \in [0, 2\pi]} \|F\|_{\Omega_d}, \max_{z \in [-1, 1]} \|F\|_{A_{d'}} \right\}$$

If  $F \in H(d, d')$ , we approximate  $F$  on  $S = [-1, 1] \times [0, 2\pi]$  as follows

$$(5.5) \quad F(z, e^{i\varphi}) \approx \ell_{MN}(z, \varphi) \equiv \sum_{m=-M}^M \sum_{n=-N}^N a_{mn} \psi_{mn}(z, \varphi)$$

where the  $a_{mn}$  are numbers and the  $\psi_{mn}$  are basis functions given by

$$(5.6) \quad \begin{aligned} \psi_{mn}(z, \varphi) &\equiv e^{im\varphi} \theta_n(z) \\ \theta_n(z) &\equiv f^{\frac{1}{2}}(z) (1-z^2) \operatorname{sinc} \left[ \frac{\omega(z) - nh}{h} \right] \\ \omega(z) &\equiv \log \left( \frac{1+z}{1-z} \right); \quad \operatorname{sinc} x = \frac{\sin(\pi x)}{\pi x}; \\ &h > 0. \end{aligned}$$

The numbers  $a_{mn}$  are given by



$$(5.7) \quad a_{mn} = \frac{1}{2\pi f^{\frac{1}{2}}(z_n)(1-z_n^2)} \int_0^{2\pi} F(z_n, e^{i\varphi}) e^{-im\varphi} d\varphi$$

$$z_n = \tanh \frac{(nh)}{2}$$

Next, recalling the definition of  $f$  and the relation (1.2), let us set

$$(5.8) \quad \gamma_2 = \frac{1}{2} \min(\alpha_2, \beta_2), \quad \gamma = \min[(\pi d \gamma_2)^{\frac{1}{2}}, \log d']$$

Theorem 5.1: Let  $F \in H$ , let  $N = M^2$ , and let  $h = [\pi d / (\gamma_2 N)]^{\frac{1}{2}}$ . Then there exist constants  $C$ ,  $C_1$  and  $C_2$  which are independent of  $N$ , such that

$$(5.9) \quad \max_{(z, \varphi) \in S} |F(z, e^{i\varphi}) - \ell_{MN}(z, \varphi)| \leq C N^{\frac{1}{2}} e^{-\gamma N^{\frac{1}{2}}}$$

$$(5.10) \quad \max_{(z, \varphi) \in S} \left| \frac{\partial}{\partial z} F(z, \varphi) - \frac{\partial}{\partial z} \ell_{MN}(z, \varphi) \right| \leq C_1 N e^{-\gamma N^{\frac{1}{2}}}$$

$$(5.11) \quad \max_{(z, \varphi) \in S} \left| \frac{\partial}{\partial \varphi} F(z, \varphi) - \frac{\partial}{\partial \varphi} \ell_{MN}(z, \varphi) \right| \leq C_2 N^{\frac{1}{2}} e^{-\gamma N^{\frac{1}{2}}}$$

Proof: It is shown in [16] that if  $g/v \in H(\Omega_d)$ , then by taking  $h = [\pi d / (\gamma_2 N)]^{\frac{1}{2}}$  there are positive constants  $C'$  and  $C''$  such that

$$(5.12) \quad \max_{z \in [-1, 1]} \left| g(z) - \sum_{n=-N}^N \frac{f^{\frac{1}{2}}(z_n)}{v(z_n)} \theta_n(z) \right| \leq C' e^{-\gamma N^{\frac{1}{2}}}$$

and

$$(5.13) \quad \max_{z \in [-1, 1]} \left| g'(z) - \sum_{n=-N}^N \frac{f'(z_n)}{v(z_n)} \theta'_n(z) \right| \leq C'' N^{\frac{1}{2}} e^{-\gamma N^{\frac{1}{2}}}$$

Similarly, it follows from Cauchy's theorem, that if  $G \in H(\Lambda_d)$ , and if  $a_m$  is defined by

$$(5.14) \quad a_m = \frac{1}{2\pi} \int_0^{2\pi} G(e^{i\varphi}) e^{-im\varphi} d\varphi, \quad m=-M, -M+1, \dots, M,$$

then there are constants  $K'$  and  $K''$  such that for all  $\varphi \in [0, 2\pi]$ ,

$$(5.15) \quad \left| G(e^{i\varphi}) - \sum_{m=-M}^M a_m e^{im\varphi} \right| \leq K' (d')^{-M}$$

and

$$(5.16) \quad \left| \frac{\partial}{\partial \varphi} G(e^{i\varphi}) - \sum_{m=-M}^M im a_m e^{im\varphi} \right| \leq K'' (d')^{-M}.$$

On noting that  $M = N^{\frac{1}{2}}$ , the inequality (5.9) is obtained if we use (5.11) and (5.15) in thm 6.2 in [24]. Similarly, (5.10) is a consequence of (5.13) and (5.15), while (5.11) follows from (5.12) and (5.16).

Theorem 5.2 [16]. If  $vg/f^{\frac{1}{2}} \in H(\Omega)_d$ , then there exists a constant  $K$  such that

$$(5.17) \quad \left| \int_{-1}^1 g(z) \theta_n(z) dz - h \frac{v(z_n)g(z_n)}{f^{\frac{1}{2}}(z_n)\omega'(z_n)} \right| \leq K e^{-\pi d/h}$$

This theorem shows that if  $g$  is any function such that  $vg/f^{\frac{1}{2}} \in H(\Omega)_d$ , then for  $h$  sufficiently small, the sequence  $\{\theta_n\}_{n=-\infty}^{\infty}$  may be considered to be an orthogonal sequence, for practical purposes, and  $g$  may be expanded with respect to this sequence. The coefficients of this expansion take on the very simple form

$$(5.18) \quad \frac{f^{\frac{1}{2}}(z_n)}{v(z_n)} g(z_n), \quad z_n = \tanh(nh/2).$$

For purposes of numerical integration, we state the following.

Theorem 5.3 [23]: Let  $g \in H(\Omega_d)$ , and let  $|g(z)| \leq C(1-z^2)^{\alpha-1}$  on  $(-1,1)$ , where  $C_1 \alpha > 0$ . If  $N > 0$  is an integer, and  $h = (2\pi d/\alpha N)^{\frac{1}{2}}$ , then there exists a constant  $C'$ , independent of  $N$ , such that

$$(5.19) \quad \left| \int_{-1}^1 g(z) dz - h \sum_{n=-N}^N \frac{g(z_n)}{\omega'(z_n)} \right| \leq C' e^{-(2\pi d \alpha N)^{\frac{1}{2}}}$$

Finally, thm. 5.4 which follows describes the accuracy of the midordinate rule used in Secs. 7 and 8 for integrating periodic functions.

Theorem 5.4: Let  $d' > 0$  and let  $g \in H(A_{d'})$  be bounded on  $A_{d'}$ . Then there exists a constant  $C$  depending only on  $g$  such that for  $M = 1, 2, 3, \dots$ ,

$$(5.20) \quad \left| \int_0^{2\pi} g(e^{i\varphi}) d\varphi - \frac{2\pi}{M} \sum_{k=1}^M g(e^{(2k-1)\pi i/M}) \right| \leq C(d')^{-M}.$$

## 6. APPROXIMATION OF THE INCIDENT ELECTRIC FIELD

We shall make the approximation

$$(6.1) \quad \bar{E}^0(\bar{r}) \approx \bar{E}^0(\bar{r}) \equiv \sum_{m=-M}^M \sum_{n=-N}^N [\alpha'_{mn} \hat{\varphi} + \beta'_{mn} \hat{t}] \psi_{mn}, \bar{r} \in S,$$

of the incident electric field, where  $\psi_{mn}$  are defined in Sec. 5. For the sake of convenience we shall use the notations

$$(6.2) \quad d_{mn} = \frac{i^m \exp \{ik_0 z_n \cos \theta_0\} J_m(k_0 f(z_n) \sin \theta_0)}{v(z_n)},$$

As will be seen in Sec. 7, we will approximate a slightly altered field  $\bar{J}^0 \equiv v\bar{E}^0$  where  $v(z) = f(z)(1-z^2)$ . Thus

$$\bar{J}^0(\bar{r}) \approx \bar{J}^0 \equiv \sum_{m=-M}^M \sum_{n=-N}^N [\alpha_{mn} \hat{\varphi} + \beta_{mn} \hat{t}] \psi_{mn}.$$

The results of Secs. 3 and 5 show that if  $\bar{r} \in S$  then each component of  $\bar{J}^0$  is in  $H(d, d')$ . Using the formulas (5.6), we have

$$(6.3) \quad \begin{aligned} \alpha_{mn} &= \frac{f^{1/2}(z_n)}{2\pi} \int_0^{2\pi} \bar{E}^0(\bar{r}) \cdot \hat{\varphi} e^{-im\varphi} d\varphi \Big|_{z=z_n} \\ \beta_{mn} &= \frac{f^{1/2}(z_n)}{2\pi} \int_0^{2\pi} \bar{E}^0(\bar{r}) \cdot \hat{t} e^{-im\varphi} d\varphi \Big|_{z=z_n} \end{aligned}$$



If this is taken together with (3.10), we get

$$\alpha_{mn} = f_n^{1/2} [ \frac{1}{2} a_1 (d_{m+1,n} + d_{m-1,n}) + \frac{\cos \theta_0}{2i} a_2 (d_{m-1,n} - d_{m+1,n}) ] \quad (6.4)$$

$$\begin{aligned} \beta_{mn} = f_n^{1/2} \{ [ a_1 f'_n(z_n) + a_2 \sin \theta_0 ] \alpha(z_n) d_{mn} - \\ - \frac{1}{2} a_2 \cos \theta_0 \alpha(z_n) f'_n(z_n) [ d_{m+1,n} + d_{m-1,n} ] \}. \end{aligned}$$

7.(a) THE GALERKIN EQUATION FOR THE SURFACE CURRENT  
DERIVATION OF THE EQUATIONS

Rather than solve (4.11) for  $\bar{K}$ , it is numerically more convenient to define  $\bar{J}$  by

$$(7.1) \quad \bar{J}(\bar{r}) = v(z) \bar{K}(\bar{r})$$

where

$$(7.2) \quad v(z) = f(z) (1-z^2),$$

and to solve the resulting integral equation for  $\bar{J}$ . This transformation helps to take account of the unknown\* singular behavior of  $\bar{K}$  at  $z = \pm 1$ , and enables us to effectively approximate both  $\bar{J}$  and its first derivative with respect to  $z$ , by methods of Sec. 5.

The substitution (7.1) replaces (4.6) by the equation

$$(7.2) \quad \left\{ \eta z \bar{J} + v \int_S \left[ \frac{i\omega\mu}{v} \bar{J} G + \frac{\eta z}{v} (\hat{n} \times \bar{J}) \times \nabla G + \frac{1}{i\omega\epsilon} (\nabla \cdot \frac{1}{v} \bar{J}) \nabla G \right] dS + \bar{J}^0_{\tan} \right\} = 0$$

where

$$(7.3) \quad \bar{J}^0 = v \bar{E}^0.$$

\*IF  $S$  satisfies Liapunov conditions (see Sec. 4), then  $\bar{K}$  is bounded on  $S$ . It is difficult to determine the exact singular behavior of the derivatives of  $K$  at  $z = \pm 1$ .

We now make the Galerkin approximation

$$(7.4) \quad \bar{J} \approx J = \sum_{m=-M}^M \sum_{n=-N}^N [a_{mn} \hat{\varphi} + b_{mn} \hat{t}] \psi_{mn}$$

in (7.2), where the  $\psi_{mn}$  are defined as in (5.16), and the  $a_{mn}$  and  $b_{mn}$  are unknown numbers.

Let us now recall that if  $\bar{r}, \bar{r}' \in S$ , then

$$(7.5) \quad R = |\bar{r} - \bar{r}'| = \{f^2 + f^{*2} - 2ff^* \cos(\varphi - \varphi') + (z - z')^2\}^{\frac{1}{2}}$$

$$G = \frac{1}{4\pi} \frac{e^{ik_0 R}}{R}$$

where here and henceforth

$$(7.6) \quad f \equiv f(z), \quad f^* \equiv f(z'), \quad \alpha = \alpha(z), \quad \alpha^* = \alpha(z').$$

We shall also use the notations

$$(7.7) \quad \left\{ \begin{array}{l} R^f = \frac{\partial R}{\partial f} = \frac{f - f^* \cos(\varphi - \varphi')}{R} \\ R^z = \frac{\partial R}{\partial z} = \frac{z - z'}{R} \\ R^\varphi = \frac{ff^* \sin(\varphi - \varphi')}{R} \\ (G^f, G^z, G^\varphi) = (R^f, R^z, R^\varphi) \frac{d}{dR} G. \end{array} \right.$$

Moreover, writing  $\sum_{m,n}$  for  $\sum_{m=-M}^M \sum_{n=-N}^N$ , we have

$$(7.8) \quad \hat{n} \times J = \sum_{m,n} \begin{vmatrix} \hat{\varphi} & \hat{t} & \hat{n} \\ 0 & 0 & 1 \\ a_{mn} & b_{mn} & 0 \end{vmatrix} \psi_{mn}$$

$$= \sum_{m,n} [ -b_{mn} \hat{\varphi} + a_{mn} \hat{t} ] \psi_{mn},$$

and also, that

$$(7.9) \quad \nabla G = \frac{1}{f} G^\varphi \hat{\varphi} + \alpha(f'G^f + G^z) \hat{t} + \alpha(G^f - f'G^z) \hat{n}$$

$$\nabla \cdot \left( \frac{1}{v} J \right) = \frac{1}{f} \sum_{m,n} \left[ \frac{a_{mn}}{v} \frac{\partial \psi_{mn}}{\partial \varphi} + \alpha b_{mn} \frac{\partial}{\partial z} \left( \frac{f}{v} \psi_{mn} \right) \right].$$

Using these results, we get

$$(7.10) \quad (\hat{n} \times J) \times \nabla G = \sum_{m,n} \begin{vmatrix} \hat{\varphi} & \hat{t} & \hat{n} \\ -b_{mn} & a_{mn} & 0 \\ \frac{1}{f} G^\varphi & \alpha(f'G^f + G^z) & \alpha(G^f - f'G^z) \end{vmatrix} \psi_{mn}$$

$$= \sum_{m,n} [ a_{mn} \alpha(G^f - f'G^z) \hat{\varphi} + b_{mn} \alpha(G^f - f'G^z) \hat{t} \\ - \{ a_{mn} \cdot \frac{1}{f} G^\varphi + b_{mn} \alpha(f'G^f + G^z) \} \hat{n} ] \psi_{mn},$$

as well as

$$(7.11) \quad \nabla \cdot \left( \frac{1}{v} J \right) \nabla G$$

$$= \left\{ \frac{1}{f} G^\varphi \hat{\varphi} + \alpha(f'G^f + G^z) \hat{t} + \alpha(G^f - f'G^z) \hat{n} \right\} \cdot \frac{1}{f} \sum_{m,n} \left[ \frac{a_{mn}}{v} \frac{\partial \psi_{mn}}{\partial \varphi} + \alpha b_{mn} \frac{\partial}{\partial z} \left( \frac{f}{v} \psi_{mn} \right) \right].$$



Substituting (7.4), (7.10), (7.11) as well as the approximation

$\bar{J}^0$

$$(7.12) \quad \bar{J}^0 \approx J^0 \equiv \sum_{m,n} [ \alpha_{mn} \hat{\varphi} + \beta_{mn} \hat{t} ] \psi_{mn}$$

into (7.2), and recalling that

$$(7.13) \quad dS = \frac{f}{\alpha} d\varphi dz,$$

we get

$$(7.14) \quad \begin{aligned} & \{ \frac{1}{2} \eta Z \sum_{m,n} [ a_{mn} \hat{\varphi}^* + b_{mn} \hat{t}^* ] \psi_{mn}^* \\ & + v^* \int_{-1}^1 \int_0^{2\pi} \{ \frac{i\omega \mu f}{\alpha v} G \sum_{m,n} [ a_{mn} \hat{\varphi} + b_{mn} \hat{t} ] \psi_{mn} \\ & + \frac{\eta Z}{v} \sum_{m,n} [ a_{mn} f(G^f - f'G^Z) \hat{\varphi} + b_{mn} f(G^f - f'G^Z) \hat{t} \\ & - \{ a_{mn} \frac{1}{\alpha} G^\varphi + b_{mn} f(f'G^f + G^Z) \} \hat{n} ] \psi_{mn} \\ & + \frac{1}{i\omega \epsilon} \{ \frac{1}{\alpha f} G^\varphi \hat{\varphi} + (f'G^f + G^Z) \hat{t} + (G^f - f'G^Z) \hat{n} \} \\ & \cdot \sum_{m,n} [ \frac{a_{mn}}{v} \frac{\partial \psi_{mn}}{\partial \varphi} + \alpha b_{mn} \frac{\partial}{\partial z'} (\frac{f}{v} \psi_{mn}) ] \} d\varphi dz \\ & + \sum_{m,n} [ \alpha_{mn} \hat{\varphi}^* + \beta_{mn} \hat{t}^* ] \psi_{mn}^* \} = 0 \end{aligned}$$

tan

where the starred variables denote functions of  $z'$  and  $\varphi'$ .

It is convenient to introduce several identities in order to reduce (7.14) to a system of linear algebraic equations. The equations (2.4) yield the identities

$$\begin{aligned}
 \hat{\varphi} \cdot \hat{\varphi}^* &= \cos(\varphi - \varphi') \\
 \hat{t} \cdot \hat{\varphi}^* &= -\alpha(z) f'(z) \sin(\varphi' - \varphi) \\
 \hat{n} \cdot \hat{\varphi}^* &= -\alpha(z) \sin(\varphi' - \varphi) \\
 \hat{\varphi} \cdot \hat{t}^* &= \alpha(z') f'(z') \sin(\varphi' - \varphi) \\
 \hat{t} \cdot \hat{t}^* &= \alpha(z') \alpha(z) [1 + f'(z) f'(z') \cos(\varphi - \varphi')] \\
 \hat{n} \cdot \hat{t}^* &= \alpha(z') \alpha(z) [f'(z') \cos(\varphi - \varphi') - f'(z)] .
 \end{aligned}
 \tag{7.15}$$

These identities are useful for taking components of  $\hat{\varphi}^*$  and  $\hat{t}^*$  in (7.14).

The identities (7.16) - (7.23) which follow serve to achieve further simplicity by enabling us to symbolically eliminate the integrations with respect to  $\varphi$ .

Upon setting

$$G(\bar{r}, \bar{r}') = \frac{e^{ik_0 |\bar{r} - \bar{r}'|}}{4\pi |\bar{r} - \bar{r}'|} \equiv \sum_{m=-\infty}^{\infty} G_m e^{im(\varphi' - \varphi)}
 \tag{7.16}$$

it follows that

$$e^{im\varphi'} G_m = \frac{1}{2\pi} \int_0^{2\pi} G(\bar{r}, \bar{r}') e^{im\varphi} d\varphi .
 \tag{7.17}$$

Notice furthermore, that  $G_{-m} = G_m$ . Therefore

$$(7.18) \quad \frac{1}{2\pi} \int_0^{2\pi} G(\bar{r}, \bar{r}') e^{im\varphi} \cos(\varphi' - \varphi) d\varphi = \frac{e^{im\varphi'}}{2} [G_{m+1} + G_{m-1}]$$

and

$$(7.19) \quad \frac{1}{2\pi} \int_0^{2\pi} G(\bar{r}, \bar{r}') e^{im\varphi} \sin(\varphi - \varphi') d\varphi = \frac{e^{im\varphi'}}{2i} [G_{m+1} - G_{m-1}].$$

By means of integration by parts, we furthermore find that

$$(7.20) \quad \begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} e^{im\varphi} \frac{\partial G(\bar{r}, \bar{r}')}{\partial \varphi} d\varphi &= \frac{-im}{2\pi} \int_0^{2\pi} e^{im\varphi} G(\bar{r}, \bar{r}') d\varphi \\ &= -im e^{im\varphi'} G_m, \end{aligned}$$

$$(7.21) \quad \begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} e^{im\varphi} \cos(\varphi - \varphi') \frac{\partial G}{\partial \varphi} d\varphi \\ = -\frac{i}{2} e^{im\varphi'} [(m+1) G_{m+1} + (m-1) G_{m-1}], \end{aligned}$$

$$(7.22) \quad \begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} e^{im\varphi} \sin(\varphi - \varphi') \frac{\partial G(\bar{r}, \bar{r}')}{\partial \varphi} d\varphi \\ = -\frac{e^{im\varphi'}}{2} [(m+1) G_{m+1} - (m-1) G_{m-1}]. \end{aligned}$$

We use (7.13) in (7.12) to take components of  $\hat{\varphi}$  and  $\hat{t}$ . Then, recalling that

$$(7.23) \quad \psi_{mn}(z, \varphi) = e^{im\varphi} \theta_n(z)$$

where

$$(7.24) \quad \theta_n(z) = \frac{v(z)}{f(z)^{1/2}} \operatorname{sinc} \left[ \frac{\omega(z) - nh}{h} \right]$$

and where  $v(z)$  is given in (7.2), and

$$(7.25) \quad \omega(z) = \log \left( \frac{1+z}{1-z} \right),$$

we can symbolically carry out the integrations with respect to  $\varphi'$  in the equation resulting from (7.12), by using (7.16) - (7.21). Upon equating coefficients of  $e^{im\varphi}$  in the resulting equations, we obtain

$$(7.26) \quad \begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \sum_n a_{mn} \theta_n + v \sum_{n=-N}^N [P^{mn} a_{mn} + Q^{mn} b_{mn}] &= - \sum_{n=-N}^N \alpha_{mn} \theta_n \\ \frac{1}{2\pi} \int_0^{2\pi} \sum_n b_{mn} \theta_n + v \sum_{n=-N}^N [R^{mn} a_{mn} + S^{mn} b_{mn}] &= - \sum_{n=-N}^N \beta_{mn} \theta_n \end{aligned}$$

$m = -M, -M+1, \dots, M.$

The coefficients  $P^{mn}$ ,  $Q^{mn}$ ,  $R^{mn}$  and  $S^{mn}$  depend on  $z'$ , and are given by



$$\begin{aligned}
 P^{mn} = \pi \int_{-1}^1 & \frac{\theta}{v} \left\{ G_{m+1} \left[ \frac{i\omega\mu f}{\alpha} + (m+1)\eta Z + \frac{m(m+1)}{i\omega\epsilon\alpha f} \right] \right. \\
 & + G_{m-1} \left[ \frac{i\omega\mu f}{\alpha} - (m-1)\eta Z + \frac{m(m-1)}{i\omega\epsilon\alpha f} \right] \\
 & + G_{m+1}^f \left[ \eta Z f + \frac{m}{i\omega\epsilon\alpha} \right] \\
 & + G_{m-1}^f \left[ \eta Z f - \frac{m}{i\omega\epsilon\alpha} \right] \\
 & \left. - (G_{m+1}^z + G_{m-1}^z)\eta Z f f' \right\} dz
 \end{aligned}
 \tag{7.27}$$

(7.28)

$$\begin{aligned}
 Q^{mn} = \pi \int_{-1}^1 & \left[ \frac{\theta}{v} \left\{ (G_{m+1} - G_{m-1}) \omega\mu f f' \right. \right. \\
 & \left. \left. - (G_{m+1}^z - G_{m-1}^z) \frac{\eta Z f}{i\alpha} \right\} \right. \\
 & + \left( \frac{f\theta}{v} \right)' \left\{ - \frac{1}{\omega\epsilon f} \left[ (m+1) G_{m+1} + (m-1) G_{m-1} \right] \right. \\
 & \left. \left. - \frac{1}{\omega\epsilon} (G_{m+1}^f - G_{m-1}^f) \right\} \right] dz
 \end{aligned}$$

$$\begin{aligned}
 R^{mn} = \pi \int_{-1}^1 \frac{\theta}{v} \{ & G_{m+1} \alpha^* f^{*'} [ \frac{-\omega \mu f}{\alpha} + i(m+1)\eta Z + \frac{m(m+1)}{\omega \epsilon \alpha f} ] \\
 & + G_{m-1} \alpha^* f^{*'} [ \frac{\omega \mu f}{\alpha} + i(m-1)\eta Z - \frac{m(m-1)}{\omega \epsilon \alpha f} ] \\
 & + G_m \alpha^* [ -2i\eta f' \eta Z ] \\
 & + G_{m+1}^f \alpha^* f^{*'} [ \frac{-\eta Z f}{i} + \frac{m}{\omega \epsilon \alpha} ] \\
 & + G_{m-1}^f \alpha^* f^{*'} [ \frac{\eta Z f}{i} + \frac{m}{\omega \epsilon \alpha} ] \\
 & + \alpha^* f^{*'} \frac{\eta Z f f'}{i} [ G_{m+1}^z - G_{m-1}^z ] \\
 & + \frac{2m\alpha^*}{\omega \epsilon \alpha} G_m^z \} dz
 \end{aligned}
 \tag{7.29}$$

$$\begin{aligned}
 S^{mn} = \pi \int_{-1}^1 \left[ \frac{\theta}{v} \{ & \alpha^* f^{*'} i\omega \mu f f' [ G_{m+1} + G_{m-1} ] + 2\alpha^* i\omega \mu f G_m \right. \\
 & + 2\eta \frac{Z f \alpha^*}{\alpha} G_m^f - \eta \frac{Z f \alpha^* f^{*'}}{\alpha} [ G_{m+1}^z + G_{m-1}^z ] \} \\
 & + \left( \frac{f\theta}{v} \right)' \{ \frac{\alpha^* f^{*'}}{i\omega \epsilon f} [ (m+1) G_{m+1} - (m-1) G_{m-1} ] \\
 & + \frac{\alpha^* f^{*'}}{i\omega \epsilon} [ G_{m+1}^f + G_{m-1}^f ] \\
 & \left. + \frac{2\alpha^*}{i\omega \epsilon} G_m^z \} \right] dz
 \end{aligned}
 \tag{7.30}$$

In these equations  $f^{*'} = f'(z')$ ,  $\alpha^* = [1 + (f^{*'})^2]^{-1/2}$ .

Next, setting  $z' = z_\ell = \tanh(\ell h/2)$  in (7.26) and using the relations

$$\theta_n(z_\ell) = \begin{cases} 0 & \text{if } n \neq \ell \\ \frac{v(z_\ell)}{f^{1/2}(z_\ell)} & \text{if } n = \ell \end{cases}
 \tag{7.31}$$

we arrive at the system

$$(7.32) \quad \begin{aligned} \frac{1}{2} \eta z a_{ml} + f^{\frac{1}{2}}(z_l) \sum_{n=-N}^N [P_l^{mn} a_{mn} + Q_l^{mn} b_{mn}] &= -\alpha_{ml} \\ \frac{1}{2} \eta z b_{ml} + f^{\frac{1}{2}}(z_l) \sum_{n=-N}^N [R_l^{mn} a_{mn} + S_l^{mn} b_{mn}] &= -\beta_{ml} \end{aligned}$$

where we have used the notation

$$(7.33) \quad \begin{aligned} P_l^{mn} &= P_l^{mn}(z_l), \quad Q_l^{mn} = Q_l^{mn}(z_l) \\ R_l^{mn} &= R_l^{mn}(z_l), \quad S_l^{mn} = S_l^{mn}(z_l) \end{aligned}$$

The system (7.32) is a block diagonal system of the form

$$(7.34) \quad \begin{bmatrix} B_{-M} & & & \\ & B_{-M+1} & & \\ & & \ddots & \\ & & & B_M \end{bmatrix} \begin{bmatrix} \bar{a}_{-M} \\ \bar{a}_{-M+1} \\ \vdots \\ \bar{a}_M \end{bmatrix} = \begin{bmatrix} \bar{\alpha}_{-M} \\ \bar{\alpha}_{-M+1} \\ \vdots \\ \alpha_M \end{bmatrix}$$

where each  $B_m$  is a complex matrix of order  $2(2N+1)$ , and (since  $G_{-m} = G_m$ )

$B_m = B_{-m}$ . The  $m$ 'th system

$$(7.35) \quad B_m \bar{a}_m = \bar{\alpha}_m$$

in (7.33) corresponds to all of the equations (7.35), for fixed  $m$ .

Thus if we denote by  $A_l^{mn}$  the  $2 \times 2$  matrix

$$(7.36) \quad A_{\ell}^{mn} = f^{\frac{1}{2}}(z_{\ell}) \begin{bmatrix} P_{\ell}^{mn} & Q_{\ell}^{mn} \\ R_{\ell}^{mn} & S_{\ell}^{mn} \end{bmatrix} + \frac{1}{2} z_{\ell} \delta_{n\ell} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$(7.37) \quad B_m = \begin{bmatrix} A_{-N}^{m,-N} & A_{-N}^{m,-N+1} & \dots & A_{-N}^{mN} \\ A_{-N+1}^{m,-N} & A_{-N+1}^{m,-N+1} & \dots & A_{-N+1}^{mN} \\ \dots & \dots & \dots & \dots \\ A_N^{m,-N} & A_N^{m,-N+1} & \dots & A_N^{mN} \end{bmatrix}$$

and

$$(7.38) \quad \bar{a}_m = \begin{bmatrix} a_{m,-N} \\ b_{m,-N} \\ a_{m,-N+1} \\ b_{m,-N+1} \\ \vdots \\ a_{mN} \\ b_{mN} \end{bmatrix} \quad \bar{\alpha}_m = - \begin{bmatrix} \alpha_{m,-N} \\ \beta_{m,-N} \\ \alpha_{m,-N+1} \\ \beta_{m,-N+1} \\ \vdots \\ \alpha_{mN} \\ \beta_{mN} \end{bmatrix}$$



7.(b) EVALUATION OF  $P^{mn}$ ,  $Q^{mn}$ ,  $R^{mn}$  and  $S^{mn}$

It is convenient to set

$$(7.39) \quad \theta_n / v = \gamma_n, \quad \left( \frac{\theta_n}{v} \right)' = \delta_n$$

The following integrals appear in (7.27) - (7.30).

$$\begin{aligned}
I_{mn}^{(1)} &= \int_{-1}^1 \gamma_n (f/\alpha) G_m dz & I_{mn}^{(4)} &= \int_{-1}^1 \gamma_n f f' G_m dz \\
I_{mn}^{(2)} &= \int_{-1}^1 \gamma_n G_m dz & I_{mn}^{(5)} &= \int_{-1}^1 \gamma_n f' G_m dz \\
I_{mn}^{(3)} &= \int_{-1}^1 [\gamma_n / (\alpha f)] G_m dz \quad m \neq 0 & I_{mn}^{(6)} &= \int_{-1}^1 \gamma_n f G_m dz \\
J_{mn}^{(1)} &= \int_{-1}^1 \gamma_n f G_m^f dz & K_{mn}^{(1)} &= \int_{-1}^1 \gamma_n f f' G_m^z dz \\
(7.40) \quad J_{mn}^{(2)} &= \int_{-1}^1 \gamma_n (1/\alpha) G_m^f dz \quad (m \neq 0) & K_{mn}^{(2)} &= \int_{-1}^1 \gamma_n (1/\alpha) G_m^z dz \\
J_{mn}^{(3)} &= \int_{-1}^1 \gamma_n (f/\alpha) G_m^f dz & K_{mn}^{(3)} &= \int_{-1}^1 \gamma_n (f/\alpha) G_m^z dz \\
L_{mn}^{(1)} &= \int_{-1}^1 \delta_n (1/f) G_m dz \quad (m \neq 0) \\
M_{mn}^{(1)} &= \int_{-1}^1 \delta_n G_m^f dz \\
N_{mn}^{(1)} &= \int_{-1}^1 \delta_n G_m^{z'} dz
\end{aligned}$$

The notations (7.40) enable us to express (7.29) - (7.30) in the "more computable" form

$$\begin{aligned}
 P^{mn} = \pi \{ & i\omega\mu (I_{m+1,n}^{(1)} + I_{m-1,n}^{(1)}) + \eta Z [(m+1) I_{m+1,n}^{(2)} - (m-1) I_{m-1,n}^{(2)}] \\
 & + \frac{m}{i\omega\epsilon} [(m+1) I_{m+1,n}^{(3)} + (m-1) I_{m-1,n}^{(3)}] \\
 & + \eta Z (J_{m+1,n}^{(1)} + J_{m-1,n}^{(1)} - K_{m+1,n}^{(1)} - K_{m-1,n}^{(1)}) \\
 & + \frac{m}{i\omega\epsilon} (J_{m+1,n}^{(2)} - J_{m-1,n}^{(2)}) \}
 \end{aligned}
 \tag{7.41}$$

$$\begin{aligned}
 Q^{mn} = \pi \{ & \omega\mu (I_{m+1,n}^{(4)} - I_{m-1,n}^{(4)}) - \frac{\eta Z}{i} (K_{m+1,n}^{(3)} - K_{m-1,n}^{(3)}) \\
 & - \frac{1}{\omega\epsilon} [(m+1) L_{m+1,n}^{(1)} + (m-1) L_{m-1,n}^{(1)} + M_{m+1,n}^{(1)} - M_{m-1,n}^{(1)}]
 \end{aligned}
 \tag{7.42}$$

$$\begin{aligned}
 R^{mn} = \pi \alpha^* f^* \{ & -\omega\mu (I_{m+1,n}^{(1)} - I_{m-1,n}^{(1)}) + i\eta Z [(m+1) I_{m+1,n}^{(2)} + (m-1) I_{m+1,n}^{(2)}] \\
 & + \frac{m}{\omega\epsilon} [(m+1) I_{m+1,n}^{(3)} - (m-1) I_{m-1,n}^{(3)}] - \frac{2i\eta Z m}{f^*} I_{mn}^{(5)} \\
 & - \frac{\eta Z}{i} (J_{m+1,n}^{(1)} - J_{m-1,n}^{(1)} - K_{m+1,n}^{(1)} + K_{m-1,n}^{(1)}) \\
 & + \frac{m}{\omega\epsilon} (J_{m+1,n}^{(2)} + J_{m-1,n}^{(2)} + \frac{2}{f^*} K_{mn}^{(2)}) \}
 \end{aligned}
 \tag{7.43}$$

and

$$\begin{aligned}
 S^{mn} = & \pi \alpha^* f^{*'} \left\{ i\omega\mu \left( I_{m+1,n}^{(4)} + I_{m-1,n}^{(4)} + \frac{2}{f^{*'}} I_{mn}^{(6)} \right) \right. \\
 & - Z \left( K_{m+1,n}^{(3)} + K_{m-1,n}^{(3)} - \frac{2}{f^{*'}} J_{mn}^{(3)} \right) \\
 (7.44) \quad & = + \frac{1}{i\omega\epsilon} \left[ (m+1) L_{m+1,n}^{(1)} - (m-1) L_{m-1,n}^{(1)} + \right. \\
 & \left. + M_{m+1,n}^{(1)} + M_{m-1,n}^{(1)} + \frac{2}{f^{*'}} N_{mn}^{(1)} \right] \}.
 \end{aligned}$$

The integrals (7.40) can be simultaneously evaluated for all  $m, n$  by evaluating the quantities  $f, f', \alpha = (1+f'^2)^{-1/2}, f^{*'} \text{ and } \alpha^* = (1+f^{*'}^2)^{-1/2}, \theta_n$  ( $n = -N, -N+1, \dots, N$ ) and  $G_m$  ( $m = -M, -M+1, \dots, N$ ). The integrals  $I_{mn}^{(1)}$  and  $L_{mn}^{(1)}$  are evaluated simultaneously by means of the formulas

$$\begin{aligned}
 (7.45) \quad \int_{-1}^1 H(z, z') dz &= \int_{-1}^{z'} H(z, z') dz + \int_{z'}^1 H(z, z') dz \\
 &= \sum_s u_s H(z', \mu_s) + \sum_s v_s H(z', \nu_s)
 \end{aligned}$$

The definitions for the exact form of  $u_s$  and  $v_s$  follow in (7.48).

On the other hand the integrals  $J_{mn}, K_{mn}, M_{mn}$  and  $N_{mn}$  are evaluated by means of the following formulas.

$J_{mn}, M_{mn}$  Here we use the results (A.2).

$$\begin{aligned}
 (7.46) \quad & \int_{-1}^1 G_m^f(z, z') g(z) dz \\
 &= - \frac{1}{4\pi^2} \cdot \frac{1}{f^*} f^* \alpha^{*2} g(z') \log \left( \frac{1-z'}{1+z'} \right) + \\
 &+ \int_{-1}^1 \left[ G_m^f(z, z') g(z) + \frac{1}{4\pi^2} \cdot \frac{f^{*'}}{f^*} \frac{\alpha^{*2}}{z-z'} g(z') \right] dz
 \end{aligned}$$



and the integral on the extreme right (7.46) is evaluated by method (7.45).

$K_{mn}, N_{mn}$ . Using (A.2),

$$(7.47) \quad \int_{-1}^1 G_m^z(z, z') g(z) dz = -\frac{1}{4\pi^2} \cdot \frac{\alpha^{*2}}{f^*} g(z') \log \left( \frac{1-z'}{1+z'} \right) \\ + \int_{-1}^1 \left[ G_m^z(z, z') g(z) + \frac{1}{4\pi^2} \frac{\alpha^{*2}}{f^*} \cdot \frac{1}{z-z'} g(z') \right] dz,$$

and the integral on the right hand side is again evaluated by means of (7.45).

We split the integration in (7.47) since  $G_m(z, z')$  has a singularity at  $z = z'$ . The formula (5.18) is then used to evaluate each integral. Thus

$$(7.48) \quad u_s = \frac{z'+1}{2} h^* \operatorname{sech}^2 \left( \frac{sh^*}{2} \right), \quad \mu_s = \frac{z'+1}{2} + \frac{z'+1}{2} \tanh \left( \frac{sh^*}{2} \right) \\ v_t = \frac{1-z'}{2} h^* \operatorname{sech}^2 \left( \frac{sh^*}{2} \right), \quad v_t = \frac{1+z'}{2} + \frac{1-z'}{2} \tanh \left( \frac{th^*}{2} \right)$$

for suitably chosen  $h^* > 0$ .

### 8. APPROXIMATION OF THE SCATTERED FIELD

The scattered field  $\bar{E}^s$  is expressed in terms of  $\bar{K}$  by means of the integral (4.7). Once  $\bar{J}$  has been obtained by means of Secs. 7, 8 and 9, we form an approximation  $\bar{K}$  of  $K$  by means of the equation (see Eq. (7.1))

$$(8.1) \quad \bar{K} = \bar{K}(z, \varphi) = \frac{1}{v(z)} \bar{J}(z, \varphi),$$

and we substitute  $\bar{K}$  for  $K$  in (4.12) to get an expression for an approximation  $\bar{E}^s$  of  $E^s$ . In this section we shall give a detailed description of the evaluation of  $\bar{E}^s$ .

We shall approximate  $\bar{E}^s(\bar{r})$  for

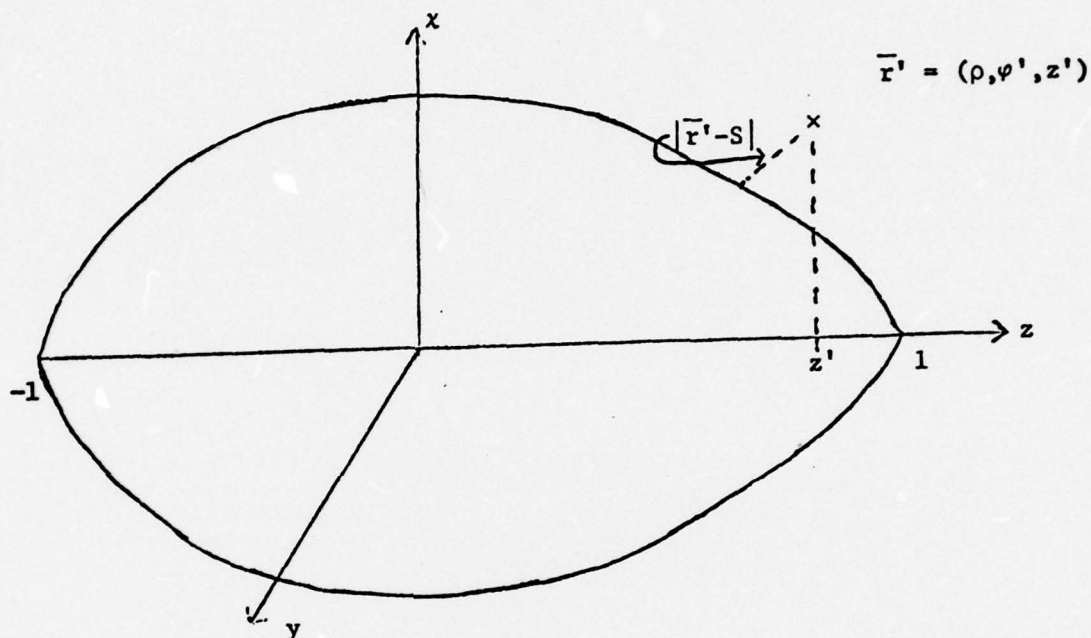
$$(8.2) \quad \bar{r}' = \rho \cos \varphi' \hat{x}' + \rho \sin \varphi' \hat{y}' + z' \hat{z}'$$

where  $\rho > f(z')$ . If  $\rho$  is close to  $f(z)$  i.e., if  $\bar{r}'$  is close to the surface, (say  $|\bar{r}' - S| \leq .2$  if  $N = 10$ ) then we recommend the integration methods of Sec. 9 to evaluate the integrals. This would involve splitting the integrals from  $-1$  to  $1$  into integrals from  $-1$  to  $z'$  and from  $z'$  to  $1$ , as in (7.45). For sake of simplicity, we shall describe an algorithm for evaluating  $\bar{E}^s$  which is valid if  $\bar{r}'$  is not arbitrarily close to  $S$  (say  $|\bar{r}' - S| > .2$  if  $N \geq 10$ ). In this latter case it is convenient to integrate by parts in the integrals with respect to  $z$  which involve  $\theta'_n$ , so that the resulting integrals involve  $\theta_n$ . This

latter procedure enables us to avoid numerical integration, by means of the approximation

$$(8.3) \quad \int_{-1}^1 H(z) \omega_n(z) dz \approx h \frac{H(z_n)}{f^{\frac{1}{2}}(z_n) \omega'(z_n)}$$

which we know to be accurate, by Thm. 5.2, where  $\omega_n = \theta_n/v$ .



Using (4.7), the scattered field may be expressed in the form

$$(8.4) \quad \overline{E}^s(\overline{r}) = E_x^s \hat{x} + E_y^s \hat{y} + E_z^s \hat{z}$$

where  $E_x^s$ ,  $E_y^s$  and  $E_z^s$  are scalar quantities. Upon substituting the approximation (7.4) into (4.7) we get

$$(8.5) \quad E_x^s \approx \sum_{m,n} [ a_{mn} P^{mn} + b_{mn} Q^{mn} ]$$

$$(8.6) \quad E_y^s \approx \sum_{m,n} [ a_{mn} R^{mn} + b_{mn} S^{mn} ]$$

$$(8.7) \quad E_z^s \approx \sum_{m,n} [ a_{mn} T^{mn} + b_{mn} U^{mn} ]$$

The relations (7.7) - (7.11) and (7.15) - (7.22) enable us to obtain explicit expression for  $P^{mn}, \dots, U^{mn}$ . Setting

$$(8.8) \quad \omega_n = \omega_n(z) = \frac{\theta_n(z)}{v(z)},$$

we get



$$\begin{aligned}
\bar{E}^s = & \int_{-1}^1 \int_0^{2\pi} \left\{ \frac{i\omega\mu}{\alpha} fG \sum_{m,n} [ a_{mn} (-\sin\varphi \hat{x} + \cos\varphi \hat{y}) \right. \\
& + b_{mn} (f' \alpha \cos\varphi \hat{x} + f' \alpha \sin\varphi \hat{y} + \alpha \hat{z}) ] e^{im\varphi} \omega_n \\
& + \eta Z \sum_{m,n} [ a_{mn} f(G^f - f'G^z) (-\sin\varphi \hat{x} + \cos\varphi \hat{y}) \\
& + b_{mn} f(G^f - f'G^z) (f' \alpha \cos\varphi \hat{x} + f' \alpha \sin\varphi \hat{y} + \alpha \hat{z}) \\
& - \{ a_{mn} \frac{1}{\alpha} G^\varphi (\alpha \cos\varphi \hat{x} + \alpha \sin\varphi \hat{y} - f' \alpha \hat{z}) \\
(8.9) \quad & + b_{mn} f(f'G^f + G^z) (\alpha \cos\varphi \hat{x} + \alpha \sin\varphi \hat{y} - f' \alpha \hat{z}) \} ] e^{im\varphi} \omega_n \\
& + \frac{1}{i\omega\epsilon} \{ \frac{1}{\alpha f} G^\varphi (-\sin\varphi \hat{x} + \cos\varphi \hat{y}) \\
& + (f'G^f + G^z) (f' \alpha \cos\varphi \hat{x} + f' \alpha \sin\varphi \hat{y} + \alpha \hat{z}) \\
& + (G^f - f'G^z) (\alpha \cos\varphi \hat{x} + \alpha \sin\varphi \hat{y} - f' \alpha \hat{z}) \} \\
& \cdot \sum_{m,n} [ a_{mn} im \omega_n + \alpha b_{mn} (f\omega_n)' ] e^{im\varphi} \} d\varphi dz
\end{aligned}$$

Here  $G = \frac{e^{ik_0 R}}{4\pi R}$ ;  $R = \{ (z-z')^2 + f^2 + \rho^2 - 2f'\rho \cos(\varphi - \varphi') \}^{\frac{1}{2}}$ .

Collecting coefficients as indicated in (8.5) to (8.7), we get

$$\begin{aligned}
P_{mn} = & \pi e^{im\phi'} \int_{-1}^1 \left\{ \frac{-\omega\mu f}{\alpha} [ e^{i\phi'} G_{m+1}^f - e^{-i\phi'} G_{m-1}^f ] + \right. \\
(8.10) \quad & + i\eta Z [ f \{ (G_{m+1}^f - f'G_{m+1}^z) e^{i\phi'} - (G_{m-1}^f - f'G_{m-1}^z) e^{-i\phi'} \} + \\
& + (m+1) e^{i\phi'} G_{m+1} + (m-1) e^{-i\phi'} G_{m-1} ] + \\
& + \frac{m}{\omega\epsilon\alpha} [ \frac{1}{f} \{ (m+1) e^{i\phi} G_{m+1}^f - (m-1) e^{-i\phi} G_{m-1}^f \} + \\
& + \{ e^{i\phi} G_{m+1}^f + e^{-i\phi} G_{m-1}^f \} ] \} \omega_n dz
\end{aligned}$$

$$\begin{aligned}
 Q_{mn} = \pi e^{im\phi'} \int_{-1}^1 \{ & [ i\omega\mu f f' [ e^{i\phi'} G_{m+1} + e^{-i\phi'} G_{m-1} ] + \\
 & - \frac{\eta Z f}{\alpha} [ e^{i\phi'} G_{m+1}^z + e^{-i\phi'} G_{m-1}^z ] ] \omega_n + \\
 & + \frac{1}{i\omega\epsilon} [ \frac{1}{f} ( (m+1) e^{i\phi'} G_{m+1} - (m-1) e^{-i\phi'} G_{m-1} ) + \\
 & + e^{i\phi'} G_{m+1}^f + e^{-i\phi'} G_{m-1}^f ] (f\omega_n)' \} dz
 \end{aligned}
 \tag{8.11}$$

$$\begin{aligned}
 R_{mn} = \pi e^{im\phi'} \int_{-1}^1 \{ & \frac{i\omega\mu f}{\alpha} (e^{i\phi'} G_{m+1} + e^{-i\phi'} G_{m-1}) + \\
 & + \eta Z [ f(e^{i\phi'} (G_{m+1}^f - f' G_{m+1}^z) + e^{-i\phi'} (G_{m-1}^f - f' G_{m-1}^z)) + \\
 & + (m+1) e^{i\phi'} G_{m+1} - (m-1) e^{-i\phi'} G_{m-1} ] + \\
 & - \frac{im}{\alpha\omega\epsilon} [ \frac{1}{f} ( (m+1) e^{i\phi'} G_{m+1} + e^{-i\phi'} G_{m-1} ) + \\
 & + e^{i\phi'} G_{m+1}^f - e^{-i\phi'} G_{m-1}^f ] \} \omega_n dz
 \end{aligned}
 \tag{8.12}$$

$$\begin{aligned}
 S_{mn} = \pi e^{im\phi'} \int_{-1}^1 \{ & [ \omega\mu f f' (e^{i\phi'} G_{m+1} - e^{-i\phi'} G_{m-1}) + \\
 & \frac{i\eta Z f}{\alpha} [ e^{i\phi'} G_{m+1}^z - e^{-i\phi'} G_{m-1}^z ] ] \omega_n + \\
 & - \frac{1}{\omega\epsilon} [ \frac{1}{f} ( (m+1) e^{i\phi'} G_{m+1} + (m-1) e^{-i\phi'} G_{m-1} ) + \\
 & + e^{i\phi'} G_{m+1}^f - e^{-i\phi'} G_{m-1}^f ] (f\omega_n)' \} dz
 \end{aligned}
 \tag{8.13}$$

$$T_{mn} = -2\pi m e^{im\phi'} \int_{-1}^1 \{ i\eta Z f' G_m - \frac{1}{\omega\epsilon\alpha} G_m^z \} \omega_n dz
 \tag{8.14}$$

$$(8.15) \quad U_{mn} = 2\pi e^{im\phi'} \int_{-1}^1 \left\{ \left[ i\omega \mu f G_m + \frac{\eta Z f}{\alpha} G_m^f \right] \omega_n + \frac{1}{i\omega \epsilon} G_m^z (f\omega_n)' \right\} dz$$

Let us next eliminate the  $(f\omega_n)'$  terms which appear in  $Q_{mn}$ ,  $S_{mn}$  and  $U_{mn}$  above. Setting

$$(8.14) \quad J_{mn} = \int_{-1}^1 \frac{1}{f} \left( \frac{1}{f} G_m \right) (f\omega_n)' dz$$

we have, upon integration by parts,

$$(8.15) \quad J_{mn} = \frac{1}{f} G_m f\omega_n \Big|_{-1}^1 - \int_{-1}^1 \left( \frac{1}{f} G_m \right)' f\omega_n dz$$

Under the assumption made on  $S$  in Sec. 1,  $\bar{K}$  is bounded on  $S$ , and therefore it follows, upon replacing  $\omega_n$  by  $\bar{K}$  in (8.15), that the first term on the right hand side of (8.15) vanishes, provided that  $m \neq 0$  (see (A.1)). However, inspection of  $Q_{mn}$  and  $S_{mn}$  shows that we need never evaluate  $J_{mn}$  if  $m = 0$ . Thus

$$(8.16) \quad J_{mn} = \int_{-1}^1 \frac{f'}{f} G_m \omega_n dz - \int_{-1}^1 G_m' \omega_n dz, \quad m \geq 1,$$

where  $G_m' = d G_m / dz$ .

Similarly, we have, for all  $m \geq 0$ ,

$$(8.17) \quad K_{mn} \equiv \int_{-1}^1 G_m^f (f\omega_n)' dz = - \int_{-1}^1 (G_m^f)' f\omega_n dz$$

and

$$(8.18) \quad L_{mn} \equiv \int_{-1}^1 G_m^z (f\omega_n)' dz = - \int_{-1}^1 (G_m^z)' f\omega_n dz,$$

these being the only remaining terms requiring integration by parts in (8.10) to (8.15).

Hence, we make the definitions

$$(8.19) \quad \begin{aligned} I_{mn}^1 &= \int_{-1}^1 \frac{f}{\alpha} G_m \omega_n dz, \quad m \geq 0; & J_{mn}^2 &= \int_{-1}^1 \frac{1}{\alpha} G_m^f \omega_n dz, \quad m \geq 0; \\ I_{mn}^2 &= \int_{-1}^1 G_m \omega_n dz, \quad m \geq 1; & J_{mn}^3 &= \int_{-1}^1 \frac{f}{\alpha} G_m^f dz, \quad m \geq 0; \\ I_{mn}^3 &= \int_{-1}^1 \frac{1}{f\alpha} G_m \omega_n dz, \quad m \geq 1; & K_{mn}^1 &= \int_{-1}^1 ff' G_m^z \omega_n dz, \quad m \geq 0; \\ I_{mn}^4 &= \int_{-1}^1 ff' G_m \omega_n dz, \quad m \geq 0; & K_{mn}^2 &= \int_{-1}^1 \frac{f}{\alpha} G_m^z \omega_n dz, \quad m \geq 0; \\ I_{mn}^5 &= \int_{-1}^1 f' G_m \omega_n dz, \quad m \geq 1; & K_{mn}^3 &= \int_{-1}^1 \frac{1}{\alpha} G_m^z \omega_n dz, \quad m \geq 1; \\ I_{mn}^6 &= \int_{-1}^1 f G_m \omega_n dz, \quad m \geq 0; & IP_{mn} &= \int_{-1}^1 G_m' \omega_n dz, \quad m \geq 1; \\ I_{mn}^7 &= \int_{-1}^1 \frac{f'}{f} G_m \omega_n dz, \quad m \geq 1; & JP_{mn} &= \int_{-1}^1 (G_m^f)' f\omega_n dz, \quad m \geq 0; \\ J_{mn}^1 &= \int_{-1}^1 f G_m^f \omega_n dz, \quad m \geq 0; & KP_{mn} &= \int_{-1}^1 (G_m^z)' f\omega_n dz, \quad m \geq 0. \end{aligned}$$



Some of the quantities in (8.19) are not required for  $m = 0$ , since their coefficient in (8.10) to (8.15) is zero. In some cases this is fortunate, since some of the above integrals do not exist when  $m$  is taken to be zero. In terms of the above integrals (8.19) we may express the terms (8.10) to (8.15) as follows.

$$\begin{aligned}
 P_{mn} = \pi e^{im\varphi} \{ & -\omega\mu [ e^{i\varphi} I_{m+1,n}^1 - e^{-i\varphi} I_{m-1,n}^1 ] \\
 & + i\eta Z [ e^{i\varphi} (J_{m+1,n}^1 - K_{m+1,n}^1) - e^{-i\varphi} (J_{m-1,n}^1 - K_{m-1,n}^1) \\
 & + (m+1) e^{i\varphi} I_{m+1,n}^2 + (m-1) e^{-i\varphi} I_{m-1,n}^2 ] \\
 & + \frac{m}{\omega\epsilon} [ (m+1) e^{i\varphi} I_{m+1,n}^3 - (m-1) e^{-i\varphi} I_{m-1,n}^3 \\
 & + e^{i\varphi} J_{m+1,n}^2 + e^{-i\varphi} J_{m-1,n}^2 ] \} ;
 \end{aligned}
 \tag{8.20}$$

$$\begin{aligned}
 Q_{mn} = \pi e^{im\varphi} \{ & i\omega\mu [ e^{i\varphi} I_{m+1,n}^4 + e^{-i\varphi} I_{m-1,n}^4 ] \\
 & - \eta Z [ e^{i\varphi} K_{m+1,n}^2 + e^{-i\varphi} K_{m-1,n}^2 ] \\
 & + \frac{1}{i\omega\epsilon} [ (m+1) e^{i\varphi} (I_{m+1,n}^7 - IP_{m+1,n}) \\
 & - (m-1) e^{-i\varphi} (I_{m-1,n}^7 - IP_{m-1,n}) \\
 & - e^{i\varphi} JP_{m+1,n} - e^{-i\varphi} JP_{m-1,n} ] \} ;
 \end{aligned}
 \tag{8.21}$$

$$\begin{aligned}
 R_{mn} = \pi e^{im\varphi} \{ & i\omega\mu [ e^{i\varphi} I_{m+1,n}^1 + e^{-i\varphi} I_{m-1,n}^1 ] \\
 & + \eta Z [ e^{i\varphi} (J_{m+1,n}^1 - K_{m+1,n}^1) + e^{-i\varphi} (J_{m-1,n}^1 - K_{m-1,n}^1) \\
 & + (m+1) e^{i\varphi} I_{m+1,n}^2 - (m-1) e^{-i\varphi} I_{m-1,n}^2 ] \\
 & - \frac{im}{\omega\epsilon} [ (m+1) e^{i\varphi} I_{m+1,n}^3 + (m-1) e^{-i\varphi} J_{m-1,n}^3 \\
 & + e^{i\varphi} J_{m+1,n}^2 - e^{-i\varphi} J_{m-1,n}^2 ] \} ;
 \end{aligned}
 \tag{8.22}$$

$$\begin{aligned}
 S_{mn} = \pi e^{im\varphi} \{ & \omega\mu [ e^{i\varphi} I_{m+1,n}^4 - e^{-i\varphi} I_{m-1,n}^4 ] \\
 & + i\eta Z [ e^{i\varphi} K_{m+1,n}^2 - e^{-i\varphi} K_{m-1,n}^2 ] \\
 & - \frac{1}{\omega\epsilon} [ (m+1) e^{i\varphi} (I_{m+1,n}^7 - IP_{m+1,n}) \\
 & + (m-1) e^{-i\varphi} (I_{m-1,n}^7 - IP_{m-1,n}) \\
 & - e^{i\varphi} JP_{m+1,n} + e^{-i\varphi} JP_{m-1,n} ] \} ;
 \end{aligned}
 \tag{8.23}$$

$$T_{mn} = -2\pi m e^{im\varphi} \{ i\eta Z I_{mn}^5 - \frac{1}{\omega\epsilon} K_{mn}^3 \} ;
 \tag{8.24}$$

$$U_{mn} = 2\pi e^{im\varphi} \{ i\omega\mu I_{mn}^6 + \eta Z J_{mn}^3 - \frac{1}{i\omega\epsilon} KP_{mn} \} .
 \tag{8.25}$$

Each of the integrals (8.19) may now be accurately evaluated using the one-term formula (8.3), provided that  $\bar{r}'$  is not unduly close to  $S$ . We shall assume this to be the case. We shall, however, require  $G_m$ ,  $G_m^f$ ,  $G_m^z$ ,  $dG_m/dz$ ,  $dG_m^f/dz$  and  $dG_m^z/dz$  in order to evaluate the integrals (8.19). Let us now describe the evaluation of these quantities.

To this end, let us set

$$\begin{aligned}
 c_j &= \cos \left[ \frac{2j-2}{4N+2} \pi \right] \\
 \rho_j &= \{ (z-z')^2 + \rho^2 + f^2 - 2f \rho c_j \}^{1/2} \\
 \alpha_j &= \frac{\partial \rho_j}{\partial z} = \frac{z-z'}{\rho_j} \\
 \beta_j &= \frac{\partial \rho_j}{\partial f} = \frac{f-2\rho c_j}{\rho_j} \\
 \gamma_j &= \frac{d\rho_j}{dz} = \alpha_j + f' \beta_j \\
 \delta_j &= \frac{d}{dz} \alpha_j = \frac{1}{\rho_j} + \frac{z\gamma_j}{\rho_j^2} \\
 \epsilon_j &= \frac{d}{dz} \beta_j = \frac{f'}{\rho_j} + \frac{2\rho c_j}{\rho_j^2} \gamma_j \\
 \omega_j &= \frac{ik_o}{\rho_j} - \frac{1}{\rho_j^2} \\
 \theta_j &= \frac{-ik_o}{\rho_j^2} + \frac{2}{\rho_j^3} \quad G_j^* = \frac{e^{ik_o \rho_j}}{\rho_j}
 \end{aligned}
 \tag{8.26}$$

The relations

$$R \equiv \{ (z-z')^2 + \rho^2 + f^2 - 2f \rho \cos \theta \}^{1/2}, \quad G^* \equiv \frac{e^{ik_o R}}{R}$$

$$G_m = \frac{1}{4\pi^2} \int_0^\pi G^* \cos m\theta \, d\theta$$

$$\frac{dG_m}{dz} = \frac{1}{4\pi^2} \int_0^\pi \left( \frac{ik_o}{R} - \frac{1}{R^2} \right) \frac{dR}{dz} G^* \cos m\theta \, d\theta$$

$$G_m^f = \frac{\partial G_m}{\partial f} = \frac{1}{4\pi^2} \int_0^\pi \left( \frac{ik_o}{R} - \frac{1}{R^2} \right) R^f G^* \cos m\theta \, d\theta$$

$$G_m^z = \frac{\partial G_m}{\partial z} = \frac{1}{4\pi^2} \int_0^\pi \left( \frac{ik_o}{R} - \frac{1}{R^2} \right) R^z G^* \cos m\theta \, d\theta$$

$$\begin{aligned} \frac{d}{dz} G_m^f &= \frac{1}{4\pi^2} \int_0^\pi \left\{ \left( -\frac{ik_o}{R^2} + \frac{2}{R^3} \right) R^f \frac{dR}{dz} + \left( \frac{ik_o}{R} - \frac{1}{R^2} \right) \frac{d}{dz} R^f \right. \\ &\quad \left. + \left( \frac{ik_o}{R} - \frac{1}{R^2} \right)^2 R^f \frac{dR}{dz} \right\} G^* \cos m\theta \, d\theta \end{aligned}$$

$$\begin{aligned} \frac{d}{dz} G_m^z &= \frac{1}{4\pi^2} \int_0^\pi \left\{ \left( -\frac{ik_o}{R^2} + \frac{2}{R^3} \right) R^z \frac{dR}{dz} + \left( \frac{ik_o}{R} - \frac{1}{R^2} \right) \frac{d}{dz} R^z \right. \\ &\quad \left. + \left( \frac{ik_o}{R} - \frac{1}{R^2} \right)^2 R^z \frac{dR}{dz} \right\} G^* \cos m\theta \, d\theta \end{aligned}$$

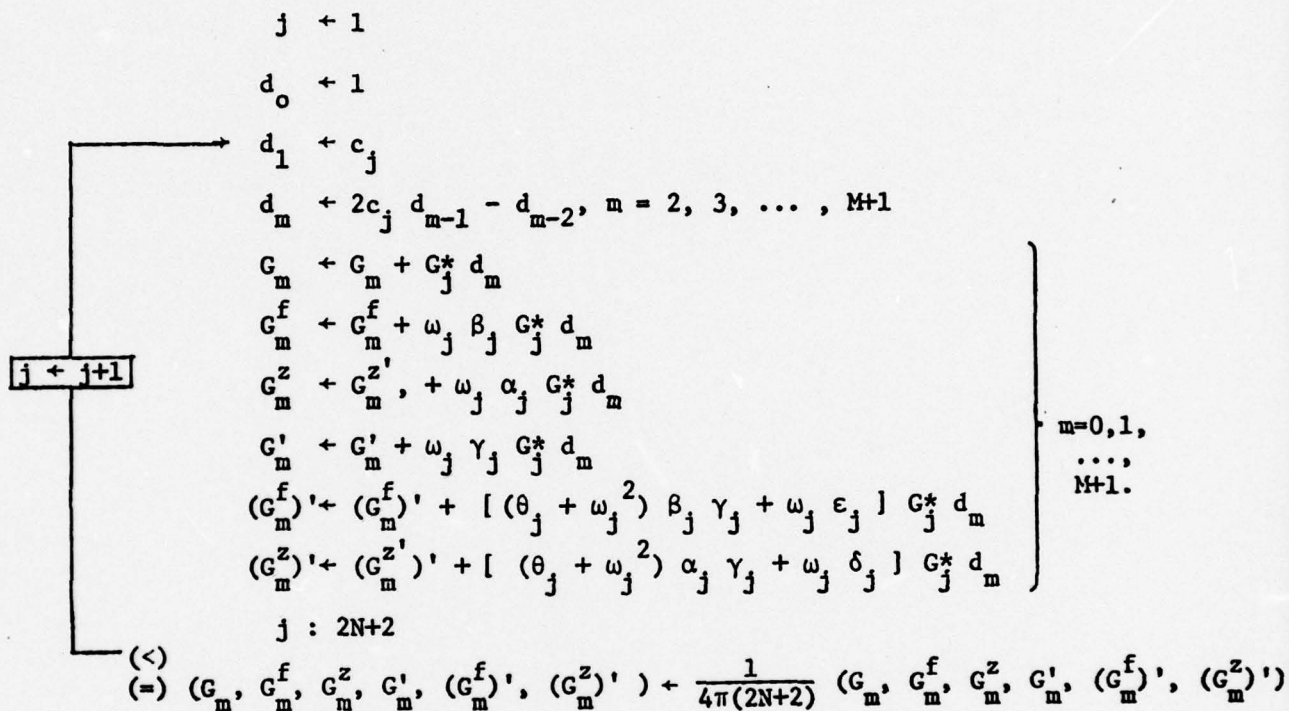
$$\cos m\theta = 2 \cos \theta \cos (m-1)\theta - \cos (m-2)\theta$$

then show that given  $z, z'$  and  $\rho$ , we can evaluate the six quantities referred to in the title of this section, for  $m = 0, 1, \dots, M+1$  by means of the following algorithm.



ALGORITHM 8.1 EVALUATION OF  $G_m, G_m^f, G_m^z, G_m', (G_m^f)', (G_m^z)'$

1. Evaluate each of the quantities (19.1), as well as  $f = f(z)$  and  $f' = f'(z)$  for  $j = 1, 2, \dots, 2N+2$ .
2.  $(G_m, G_m^f, G_m^z, (G_m)', (G_m^f)', (G_m^z)') \leftarrow (0, 0, 0, 0, 0, 0)$



9. CONVERGENCE

The proof of convergence of the approximation scheme presented in Secs. 6 to 8 is quite simple, using thm. A.1 and the results of Sec. 5.

Let us denote the right hand side of (A.36) by  $A\bar{J}_1$  and for  $F \in H(d, d')$ , let us denote  $\ell_{MN}$  by  $P_{MN}(F)$ , where  $\ell_{MN}(z, \varphi)$  is defined in (5.5).

In view of Thm. A.1, and Sec. 5, we have

$$(a) \|P_{MN} AJ - AJ\|_H \rightarrow 0 \text{ for every } J \in H(d, d');$$

$$(b) \|P_{MN} J^\circ - \bar{J}^\circ\|_H \rightarrow 0$$

$$(c) \sup \|P_{MN}\| \leq 1 + \|I - P_{MN}\| < \infty$$

Hence according to [21 pp. 469-470] it follows that the approximation  $\bar{J} = \bar{J}_{MN}$  produced by the algorithm of Sections 6 to 8 converges to the solution  $\bar{J}$  of Eq. (7.2). Moreover, by taking  $N = M^2$ , it follows that

$$(9.1) \quad \|\bar{J} - \bar{J}_{MN}\|_H = O(e^{-\gamma N^{\frac{1}{2}}})$$

for some  $\gamma > 0$ . Due to quadratures and matrix solution involved in the actual algorithm we actually compute a perturbed solution  $\tilde{J}_{MN}$ ; however, due to the accuracy of the quadrature schemes described in Sec. 5, and since the resulting Galerkin matrix is not ill-conditioned, we also have

$$(9.2) \quad \|\bar{J}_{MN} - \tilde{J}_{MN}\|_H = O(e^{-\gamma N^{\frac{1}{2}}}), \quad N \rightarrow \infty$$

By combining (9.1) and (9.2), it thus follows that

$$(9.3) \quad \|\bar{J} - \tilde{J}_{MN}\|_H = O(e^{-\gamma N^{\frac{1}{2}}}), \quad N \rightarrow \infty.$$

Finally, in computing the scattered field  $\bar{E}^s = \bar{E}_{MN}^s$ , as described in Sec. 8, we similarly have by Thm. 5.2 that

$$(9.4) \quad \|\bar{E}^s - \tilde{E}_{MN}\|_H = O(e^{-\gamma N^{\frac{1}{2}}})$$

where  $\tilde{E}_{MN}$  denotes the perturbed scattered field that we actually computed.

APPENDIX A: THE FUNCTIONS  $G_m$ ,  $\partial G_m / \partial f$  and  $\partial G_m / \partial z$ .

In this appendix we study the functions  $G_m$ ,  $G_m^f = \partial G_m / \partial f$  and  $G_m^z = \partial G_m / \partial z$  which appear in Secs. 7 to 10. The results of this study will enable us to deduce the following:

- i) Each component of the solution  $\bar{J}$  of Eq. (7.2) is in  $H(d, d')$ ;
- ii) If  $m \geq 0$ ,

$$(A.1) \quad \left\{ \begin{array}{l} G_m(z, z') = O(|f(z)|^m) \\ G_m^f(z, z') = \begin{cases} O(1) & \text{if } m = 0 \\ O(|f(z)|^{m-1}), & m > 0 \end{cases} \\ G_m^z(z, z') = O(|f(z)|^m) \end{array} \right\} \quad \begin{array}{l} \text{as } z \rightarrow \pm 1, \\ z' \in (-1, 1); \end{array}$$

and

$$(A.2) \quad \left\{ \begin{array}{l} G_m(z, z') \sim \frac{1}{4\pi^2 f(z')} \log \frac{1}{|z-z'|} \\ G_m^f(z, z') \sim \frac{-f'(z')\alpha(z')^2}{4\pi^2 f(z')(z-z')} \\ G_m^z(z, z') \sim \frac{-\alpha(z')^2}{4\pi^2 f(z')(z-z')} \end{array} \right\} \quad \begin{array}{l} \text{as } z \rightarrow z', \\ z' \in (-1, 1). \end{array}$$

The results (A.1) proved to be useful for choosing the basis functions, in order to be able to obtain a convergent Galerkin method, while the results (A.2) enabled us to choose the proper numerical



integration technique for evaluating the singular integrals to get the coefficients  $P^{mn}$ ,  $Q^{mn}$ ,  $R^{mn}$  and  $S^{mn}$  in Secs. 7-9.

Throughout this appendix, the following notation is being used.

$$\begin{aligned}
 f &= f(z) , \quad f^* = f(z') \\
 v &= \{(z-z')^2 + (f+f^*)^2\}^{\frac{1}{2}} \\
 (A.3) \quad \kappa &= \frac{4ff^*}{v^2} \\
 1-\kappa &= \frac{(f-f^*)^2 + (z-z')^2}{v^2}
 \end{aligned}$$

(a) The Functions  $G_m$

The functions  $G_m$  (see Eqs. (7.10) and (7.17) ) are defined for given  $z, z'$  on  $(-1,1)$  by

$$(A.4) \quad G_m = G_m(z, z') = \frac{1}{4\pi^2} \int_0^\pi \frac{e^{ik_o R}}{R} \cos(m\theta) d\theta,$$

where

$$(A.5) \quad R = \{(z-z')^2 + f^2 + f^{*2} - 2ff^* \cos \theta\}^{\frac{1}{2}}$$

In terms of (A.3), we therefore have

$$\begin{aligned}
 R &= v \left\{ 1 - \kappa \cos^2 \frac{\theta}{2} \right\}^{\frac{1}{2}} \\
 (A.6) \quad G_m &= \frac{1}{2\pi^2} \int_0^{\pi/2} \frac{e^{ik_o v \{1-\kappa \cos^2 \theta\}^{\frac{1}{2}}}}{v \{1-\kappa \cos^2 \theta\}^{\frac{1}{2}}} \cos(2m\theta) d\theta
 \end{aligned}$$

Expansion of the exponential in (A.6) and termwise integration yields

$$(A.7) \quad G_m = \frac{1}{2\pi^2} \sum_{s=0}^{\infty} \frac{(-1)^s v^{2s} k_o^{2s}}{(2s)!} \left\{ \frac{J_m^s}{v} + i \frac{k_o K_m^s}{2s+1} \right\}$$

where

$$(A.8) \quad J_m^s = \int_0^{\pi/2} \{1 - \kappa \cos^2 \theta\}^{s-1/2} \cos(2m\theta) d\theta$$

$$K_m^s = \int_0^{\pi/2} \{1 - \kappa \cos^2 \theta\}^s \cos(m\theta) d\theta.$$

The relationship

$$(A.9) \quad \{1 - \kappa \cos^2 \theta\}^a \cos(2m\theta)$$

$$= \{1 - \kappa \cos^2 \theta\}^{a-1} \left[ \left(1 - \frac{\kappa}{2}\right) \cos(2m\theta) - \frac{\kappa}{4} \cos(2m+2)\theta - \frac{\kappa}{4} \cos(2m-2)\theta \right]$$

shows that

$$(A.10) \quad J_m^s = \left(1 - \frac{\kappa}{2}\right) J_m^{s-1} - \frac{\kappa}{4} J_{m+1}^{s-1} - \frac{\kappa}{4} J_{m-1}^{s-1}$$

$$K_m^s = \left(1 - \frac{\kappa}{2}\right) K_m^{s-1} - \frac{\kappa}{4} K_{m+1}^{s-1} - \frac{\kappa}{4} K_{m-1}^{s-1},$$

whereas, by (A.8)

$$(A.11) \quad J_{-m}^s = J_m^s, \quad K_{-m}^s = K_m^s.$$

The relationships (A.10) show that in order to evaluate  $G_m$  using (A.7), we need only know  $J_m^0$  and  $K_m^0$ , for  $m = 0, \pm 1, \pm 2, \dots$ ; we can then get the remaining  $J_m^s$  and  $K_m^s$  for  $s > 0$  using (A.10). To this end, integrating termwise in (A.8) and using the identity:

$$(A.12) \quad \int_0^{\pi/2} \cos^{2n} \theta \cos(2m\theta) d\theta = \binom{2n}{n+m} \frac{\pi}{2^{2n+1}}$$

we get

$$(A.13) \quad J_m^0 = \frac{\pi}{2} \frac{(\frac{1}{2})_m \kappa^m}{2^{2m} m!} F\left(\frac{1}{2}+m, \frac{1}{2}+m; 1+2m; \kappa\right)$$

$$K_m^0 = \begin{cases} \frac{\pi}{2} & \text{if } m = 0 \\ 0 & \text{if } m \neq 0. \end{cases}$$

In (A.13)  $F$  denotes the hypergeometric function. Thus we may compute  $J_m^0$  for  $0 \leq \kappa \leq .6$  by means of the formula

$$(A.14) \quad J_m^0 = \frac{\pi}{2} \frac{(\frac{1}{2})_m \kappa^m}{2^{2m} m!} \sum_{n=0}^{\infty} \frac{[(\frac{1}{2}+m)_n]^2}{(1+2m)_n n!} \kappa^n$$

whereas, if  $.6 < \kappa < 1$  (see [1, p.559]) it is preferable to use the formula

$$(A.15) \quad J_m^0 = \kappa^m \sum_{n=0}^{\infty} \frac{[(\frac{1}{2}+m)_n]^2}{[n!]^2} [\psi(n+1) - \psi(\frac{1}{2}+m+n) - \frac{1}{2} \ln(1-\kappa)] (1-\kappa)^n$$

where

$$\begin{aligned}
 (A.16) \quad \psi(1) &= -\gamma = -.57721 \quad 56649 \\
 \psi(\tfrac{1}{2}) &= -\gamma - 2\ln 2 = -1.9635 \quad 10026 \\
 \psi(z+1) &= \frac{1}{z} + \psi(z).
 \end{aligned}$$

Eq. (A.15) shows that

$$(A.17) \quad J_m^0 \sim -\tfrac{1}{2} \ln(1-\kappa) \text{ as } \kappa \rightarrow 1^-.$$

Now, by (A.3)

$$\begin{aligned}
 (A.18) \quad 1-\kappa &= \frac{v^2 - 4ff^*}{v^2} = \frac{(z-z')^2 + (f-f^*)^2}{v^2} \\
 &\sim \frac{[1+f^{*'}^2](z-z')}{4f^{*'}^2} \text{ as } z \rightarrow z'
 \end{aligned}$$

so that, by combining (A.17) and (A.18), we get

$$(A.19) \quad J_m^0 \sim \ln \left[ \frac{2f^*}{(1+f^{*'}^2)^{\frac{1}{2}}} \frac{1}{|z-z'|} \right], \quad z \rightarrow z'.$$

Using induction on (A.10), it thus follows that

$$(A.20) \quad J_m^s \sim 0 \text{ as } z \rightarrow z', \quad s > 0.$$

In view of (A.7), (A.13), (A.19) and (A.20) it therefore follows that

$$\begin{aligned}
 (A.21) \quad G_m &\sim \frac{1}{2\pi^2} \frac{J_m^0}{v} \sim \frac{1}{4\pi^2 f^*} \ln \left[ \frac{2f^*}{(1+f^{*'}^2)^{\frac{1}{2}}} \frac{1}{|z-z'|} \right] \\
 &\text{as } z \rightarrow z',
 \end{aligned}$$



which is the first of (A.2).

In view of (A.3), it follows that

$$(A.22) \quad \kappa = O(f) \text{ as } z \rightarrow \pm 1,$$

for all  $z' \in [-1, 1]$ . Hence by (A.13) and (A.14),

$$(A.23) \quad G_m \sim O(f^{|m|}) \text{ as } z \rightarrow \pm 1, \text{ for all } z' \in [-1, 1].$$

$$(b) \quad \text{The Functions } G_m^f, \quad G_m^{z'}$$

Upon differentiating the expressions (A.3) we get

$$(A.24) \quad \begin{aligned} v^f &\equiv \frac{\partial v}{\partial f} = \frac{f+f^*}{v}, \quad v^z \equiv \frac{\partial v}{\partial z} = \frac{z-z'}{v} \\ \kappa^f &= \frac{\partial \kappa}{\partial f} = \frac{4f^*}{v^2} - \frac{8ff^*(f+f^*)}{v^4} \\ \kappa^z &= \frac{\partial \kappa}{\partial z} = -\frac{8ff^*}{v^3} \frac{(z-z')}{v} \end{aligned}$$

We note therefore, that

$$(A.25) \quad \begin{aligned} 2 v^f + v \frac{\kappa^f}{\kappa} &= \frac{\gamma}{f} \\ 2 v^z + v \frac{\kappa^z}{\kappa} &= 0 \end{aligned}$$

By means of these solutions as well as (A.10), we get the identities

$$\begin{aligned}
 G_m^f &\equiv \frac{\partial G_m}{\partial f} = \frac{1}{2\pi^2} \sum_{s=0}^{\infty} \left\{ \frac{(-1)^s v^{2s} k_o^{2s}}{(2s)!} \left[ \frac{s-1/2}{v} \left( \frac{1}{f} J_m^s - \frac{k^f}{K} J_m^{s-1} \right) \right. \right. \\
 &\quad \left. \left. - \frac{i v^2 k_o^3}{2(2s+1)(2s+3)} \left( \frac{1}{f} K_m^{s+1} - \frac{K^f}{K} K_m^s \right) \right] \right\}
 \end{aligned}
 \tag{A.26}$$

and

$$G_m^z = \frac{z-z'}{\pi^2 v^3} \sum_{s=0}^{\infty} \left\{ \frac{(-1)^s v^{2s} k_o^{2s}}{(2s)!} \left[ (s-1/2) J_m^{s-1} + \frac{-i v^2 k_o^3}{2(2s+1)(2s+3)} K_m^s \right] \right\}.
 \tag{A.27}$$

These series may be readily computed using the identities

$$\begin{aligned}
 J_m^{-1} &= \frac{\pi}{2} \frac{(3/2)_m K^m}{2^m m!} F\left(\frac{3}{2} + m, \frac{1}{2} + m; 2m + 1; \kappa\right) \\
 &= \frac{\pi}{2} \frac{(3/2)_m K^m}{2^m m!} \sum_{n=0}^{\infty} \frac{(3/2+m)_n (\frac{1}{2}+m)_n}{(2m+1)_n n!} \kappa^n, \quad 0 \leq \kappa \leq .6 \\
 &= \frac{\kappa^m}{1-\kappa} + \kappa^m (m^2 - \frac{1}{4}) \sum_{n=0}^{\infty} \frac{(\frac{3}{2}+m)_n (\frac{1}{2}+m)_n}{n! (n+1)!} (1-\kappa)^n.
 \end{aligned}
 \tag{A.28}$$

$$\cdot \left[ \ln(1-\kappa) + 2\psi\left(\frac{1}{2}+m+n\right) - 2\psi(n+1) + \frac{1}{m+n+\frac{1}{2}} - \frac{1}{n+1} \right]$$

$$\text{if } .6 \leq \kappa < 1$$

along with (A.10). These identities were obtained via a procedure similar to that which was used to get (A.14) and (A.15).

The expressions (A.26) and (A.28) enable us to deduce various growth properties of  $G_m^f$  and  $G_m^z$  as  $z \rightarrow z'$  and as  $z \rightarrow \pm 1$ .

By (A.28) and (A.3),

$$(A.29) \quad \begin{aligned} J_m^{-1} &\sim \frac{\kappa^m}{1-\kappa} \\ &\sim \frac{4f^{*2}\alpha^{*2}}{(z-z')^2} \quad \text{as } z \rightarrow z' \in (-1,1) \end{aligned}$$

and

$$(A.30) \quad J_m^{-1} = O(f^m) \quad \text{as } z \rightarrow \pm 1$$

Combining these results with the asymptotic identities

$$(A.31) \quad \begin{cases} \frac{\kappa^f}{\kappa} \sim -\frac{f^{*'}(z-z')}{2f^{*2}}, & z \rightarrow z' \\ \sim \frac{1}{f}, & z \rightarrow \pm 1, \end{cases}$$

and

$$(A.32) \quad \frac{\kappa^z}{\kappa} \sim -\frac{2f^{*2}(z-z')}{\nu^2}, \quad z \rightarrow z'$$

$$\frac{\kappa^z}{\kappa} = O(f), \quad z \rightarrow \pm 1$$

we get

$$(A.33) \quad \begin{aligned} G_m^f &\sim -\frac{1}{4\pi^2} \frac{f^{*'}\alpha^{*2}}{f^{*}(z-z')}, \quad z \rightarrow z' \\ G_m^z &\sim -\frac{1}{4\pi^2} \frac{\alpha^{*2}}{f^{*}(z-z')}, \quad z \rightarrow z' \end{aligned}$$



The relation (A.10), (A.13), (A.26), (A.28) and (A.31) yield

$$(A.34) \quad G_m^f = \begin{cases} 0(1) & \text{if } m=0 \\ 0(f^{m-1}) & \text{if } m \neq 0 \end{cases} \quad \text{as } z \rightarrow \pm 1,$$

while the relations (A.10), (A.13), (A.27), (A.28) and (A.32) yield

$$(A.35) \quad G_m^z = 0(f^m) \quad \text{as } z \rightarrow \pm 1.$$

(c) Analyticity of  $\bar{K}$

The above results show that

- i)  $G_m$  is bounded as a function of  $z$  on  $[-1, 1]$ , except at  $z = z'$  where it becomes unbounded according to (A.21);
- ii)  $G_m(z, z')$  is an analytic function of  $z \in \Omega_d$ , except at  $z = z'$ , where it has a singularity of the form (A.21).

In the following theorem  $H(d, d')$  is defined as in Sec. 5.

Theorem A.1: Let  $\bar{K}$  be the solution of Eq. (4.6), and let  $\bar{J}$  be defined by (7.1). Then each component of  $\bar{J}$  is in  $H(d, d')$ .

Proof: Each component of the incident field  $\bar{J}^0$  (see Eq. (7.3)) is in  $H(d, d')$  where  $d' > 0$  is arbitrary. In view of Eq. (7.2), we need only show that if  $\bar{J}_1 = (J_{1t}, J_{1\varphi})$  denotes a pair of functions in  $H(d, d')$ , each of which is bounded on each of the sets

$$S_1 = \{ (z, \omega) : z \in \Omega_d, |\omega| = 1 \}$$

$$S_2 = \{ (z, \omega) : -1 \leq z \leq 1, \frac{1}{d'} \leq |\omega| \leq d' \}$$

where  $d' > 1$  is arbitrary.



then each of the components of

$$\begin{aligned}
 J_2 &= (J_{2t}, J_{2\varphi}) \\
 (A.36) \quad &= \left\{ \nu \int_S \left[ \frac{i\omega\mu}{\nu} \bar{J}_1 G + \eta \frac{Z}{\nu} (\hat{n} \times \bar{J}_1) \times \nabla G \right. \right. \\
 &\quad \left. \left. + \frac{1}{i\omega\epsilon} \nabla \cdot \left( \frac{1}{\nu} \bar{J}_1 \right) \nabla G \right] dS \right\}_{tan}
 \end{aligned}$$

is in  $H(d, d')$ .

By our assumption on  $f$  in Sec. 3, the solution  $\bar{K} = (K_t, K_\varphi)$  of Eq. (4.2) is bounded on  $S = [-1, 1] \times [0, 2\pi]$ . Hence for  $z \in [-1, 1]$ , each component of  $\bar{J}_1$  has the form

$$(A.37) \quad F(z, e^{i\varphi}) = \sum_{m=-\infty}^{\infty} a_m(z) e^{im\varphi}$$

where  $a_m/\nu \in H(\Omega_d)$ ; substituting this form of an expression into (A.36) for  $J_{1t}$  and  $J_{1\varphi}$  and noting that  $a_m/\nu = O(e^{-d'|m|})$   $\forall z \in [-1, 1]$  and for all  $d' > 0$ , we deduce, by inspection of (7.27) - (7.30) and (A.1) and (A.2) that  $|P^{mn}|$ ,  $|Q^{mn}|$ ,  $|R^{mn}|$  and  $|S^{mn}|$  are  $O(e^{-d'|m|})$  for all  $d' > 0$  and for all  $z' \in [-1, 1]$ . That is,  $F(z, \omega) \in H(A_d)$  as a function of  $\omega$ , for all  $z \in [-1, 1]$ .

Hence in order to complete the proof, we need only show that if  $\theta_n/\nu \in H(\Omega_d)$ , then each of the right-hand sides of (7.27) to (7.30) is in  $H(\Omega_d)$ .

In view of the results of Appendix A.(a) and A.(b), the coefficients of  $\theta_n/\nu$  in the integrals (7.27) to (7.30) are of the following three types:

$$a(z, z'), a(z, z') \log |z - z'|, a(z, z')/(z - z'),$$

where  $a(z, z')$  is a bounded function in  $H(\Omega_d) \times H(\Omega_d)$ . Hence, we need only show that given  $g \in H(\Omega_d)$ , each of the functions  $g_1$ ,  $g_2$  and  $g_3$  are analytic in  $\Omega_d$ , where

$$\begin{aligned} g_1(z') &= \int_{-1}^1 a(z, z') g(z) dz; \\ (A.38) \quad g_2(z') &= \int_{-1}^1 a(z, z') \log |z - z'| g(z) dz; \\ g_3(z') &= \text{P.V.} \int_{-1}^1 \frac{a(z, z')}{z - z'} g(z) dz \end{aligned}$$

It is obvious that  $g_1 \in H(\Omega_d)$ .

Next if  $z' \in \Omega_d \cap \{\text{Im } z' > 0\}$  then

$$(A.39) \quad g_3^*(z') = \int_{-1}^1 \frac{a(z, z')}{z - z'} g(z) dz$$

is analytic in this region, and indeed, by altering the path of integration in (A.39) to the lower boundary of  $\Omega_d$ , we see that  $g_3^*$  is in fact analytic in  $\Omega_d$ . If we now return the path of integration to the interval  $(-1, 1)$  and let  $\text{Im } z' \rightarrow 0$ , we find that for  $z' \in (-1, 1)$ ,

$$(A.40) \quad g_3^*(z') = \pi i a(z', z') g(z') + \pi i g_3(z')$$

this expression shows that since both  $g_3^*(z')$  and  $a(z', z')$  have an analytic extension into  $\Omega_d$ , so does  $g_3$ . Hence  $g_3$  is analytic in  $\Omega_d$ .

Finally, writing  $g_2$  in the form of a convergent sum

$$(A.41) \quad g_2(z') = \int_{-1}^1 \sum_k \alpha_k(z) \beta_k(z') \log |z-z'| dz$$

where the functions  $\alpha_n$  and  $\beta_n$  are in  $H(\Omega_d)$ , we need only show that  $g_k^*$  is analytic in  $\Omega_d$ , where

$$(A.42) \quad g_k^*(z') = \int_{-1}^1 \alpha_k(z) \log |z-z'| dz,$$

Upon differentiating this expression carefully, we see that

$$(A.43) \quad g_k^{*'}(z') = \text{P.V.} \int_{-1}^1 \frac{\alpha_k(z)}{z-z'} dz.$$

By our argument involving  $g_3$  above, it follows that  $g_k^{*'}$  is analytic  $\Omega_d$ , i.e.,  $g_k^*$  and hence  $g_2$  is analytic in  $\Omega_d$ . This completes the proof of Theorem A.1.

By assumption for the case of finite conductivity of the body B, the function  $f$  is such that the surface  $S$  satisfies Liapunov conditions, in which case the surface current  $\bar{K}$  is bounded on  $S$ . However, one notes that the integrals (7.27) - (7.30) converge so long as  $f |\theta_n/v| = o(1)$  as  $z \rightarrow \pm 1$ , i.e., so long as  $fK = o(1)$  as  $z \rightarrow \pm 1$ . Thus our method gives answers even if  $S$  is cone-shaped at one or both ends, although the Liapunov conditions are then violated, and our results may have no physical significance, except in the infinite conducting case.



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## 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A Galerkin integral equation method is described for solving the problem of the title. Each component of the scattered field is expressed in the form

$$(1) E = \sum_{m=-M}^M \sum_{n=-M^2}^{M^2} A_{mn} e^{im\phi} \frac{\sin\{\pi M(x-n/M)\}}{\pi M(x-n/M)}$$

Under the assumption that the surface of the body is described by  $\sqrt{x^2+y^2} = f(x)$  where  $f$  is analytic and bounded in  $\{z : |\arg[(1+z)/(1-z)]| < d\}$ , the rate of convergence of (1) is

20.

$0(e^{-cM})$ ; this rate cannot be improved.